Quantum computational logic with mixed states

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Abstract

In this paper we solve a problem posed in [3] and [5] about the axiomatizability of a system of quantum computational gates known as the Poincaré irreversible quantum computational system. A Hilbert-style calculus is introduced obtaining a strong completeness theorem.

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Introduction

The idea of quantum computation was introduced in 1982 by Richard Feynmann and remained primarily of theoretical interest until developments such as the invention of an algorithm to factor large numbers triggered a vast domain of research. In a classical computer, information is encoded in a series of bits and these bits are manipulated via Boolean logical gates like *NOT, OR, AND*, etc, arranged in succession to produce an end result. Standard quantum computing is based on quantum systems described by finite dimensional Hilbert spaces, specially $\mathbb{C}^2$ -the two-dimensional space of a *qbit*. A qbit (the quantum counterpart of the classical bit) is represented by a

*Fellow of the Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET) unit vector in $\mathbb{C}^2$ and, generalizing for a positive integer n, *n-qbits* are represented by unit vectors in $\mathbb{C}^{2^n}$. Similarly to the classical computing case, we can introduce and study the behavior of a number of *quantum logical gates* (hereafter quantum gates for short) operating on qbits. Quantum computing can simulate all computations which can be done by classical systems; however, one of the main advantages of quantum computation and quantum algorithms is that they can speed up computations [19].

In [3] and [5], a quantum gate system called *Poincaré irreversible quantum computational system* ($\mathbb{IP}$-system for short) was developed. The $\mathbb{IP}$-system is an interesting set of quantum gates specially for two reasons: i) it is related to continuous t-norms [17], i.e. continuous binary operations on the interval $[0,1]$ that are commutative, associative and non-decreasing with 1 is the unit element. They are naturally proposed as interpretations of the conjunction in systems of fuzzy logic [13]. ii) Subsequent generalizations allow to connect the $\mathbb{IP}$-system with sequential effect algebras [10], introduced to study the sequential action of quantum effects which are unsharp versions of quantum events [11, 12].

Our work is motivated by the $\mathbb{IP}$-system, and mainly by the following question proposed by the authors in [3] and [5]: *"The axiomatizability of quantum computational logic is an open problem."*. To answer this claim, we study an algebraic structure related to the $\mathbb{IP}$-system and we provide a Hilbert-style calculus, obtaining a strong completeness theorem with respect to the mentioned structure.

The paper is structured as follows: In Section 1, we briefly resume basic physical notions of mathematical approaches to quantum computation. Section 2 contains generalities on universal algebra and algebraic structures associated with Łukasiewicz infinite-valued calculus as MV-algebras and product MV-algebras. In Section 3 we introduce a set of quantum gates known as Poincaré irreversible quantum gates system. The mathematical representation of these quantum gates is closely related with the product MV-algebras structure. In Section 4, algebraic structures associated to quantum computation are introduced. Specifically, we introduce an expansion of the equational class known as square root quasi MV-algebras [9], expansion that we call "square root quasi PMV-algebra" (or $\sqrt{qPMV}$-algebra for short). In Section 5 we focus on a subvariety of the $\sqrt{qPMV}$-algebras called *Irreversible Poincaré Algebras.* In Section 6 we introduce the notion of probabilistic consequence. It provides a generalization of a classical problem related to digital circuits which consists in knowing whether a determinate state of the output of a set of a circuits conditions a determine state of the output of another circuit. Finally, in Section 7 we give a Hilbert-style axiomatization, called $\mathcal{LIP}$, for the probabilistic consequence. A strong completeness theorem for $\mathcal{LIP}$ with respect to the variety of Irreversible Poincaré Algebras is obtained.

1 Basic notions in quantum computation

In quantum computation, information is elaborated and processed by means of quantum systems. The pure state of a quantum system is described by a unit vector in a Hilbert space, denoted by $|\varphi\rangle$ in Dirac notation. A *quantum bit* or *qbit*, the fundamental concept of quantum computation, is a pure state in the Hilbert space $\mathbb{C}^2$. The standard orthonormal basis $\{|0\rangle, |1\rangle\}$ of $\mathbb{C}^2$ where $|0\rangle = (1,0)$ and $|1\rangle = (0,1)$ is called the *logical basis*. Thus, pure states $|\varphi\rangle$ in $\mathbb{C}^2$ are coherent superpositions of the basis vectors with complex coefficients

$$|\psi\rangle = c_0|0\rangle + c_1|1\rangle, \quad \text{with} \quad |c_0|^2 + |c_1|^2 = 1$$

Recalling the Born rule, any qubit $|\psi\rangle = c_0|0\rangle + c_1|1\rangle$ may be regarded as a piece of information, where the number $|c_0|^2$ corresponds to the probability-value of the information described by the basic state $|0\rangle$; while $|c_1|^2$ corresponds to the probability-value of the information described by the basic state $|1\rangle$. The two basis-elements $|0\rangle$ and $|1\rangle$ are usually taken as encoding the classical bit-values 0 and 1, respectively. By these means, a probability value is assigned to a qbit as follows:

Definition 1.1 [3], [5] Let $|\psi\rangle = c_0|0\rangle + c_1|1\rangle$ be a qbit. Then its *probability value* is $p(|\psi\rangle) = |c_1|^2$

The quantum states of interest in quantum computation lie in the tensor product $\otimes^n\mathbb{C}^2 = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \ldots \otimes \mathbb{C}^2$ (n times), a 2^n-dimensional complex space. A special basis, called the 2^n-*computational basis*, is chosen for $\otimes^n\mathbb{C}^2$. More precisely, it consists of the 2^n orthogonal states $|\iota\rangle$, $0 \leq \iota \leq 2^n$ where ι is in binary representation and $|\iota\rangle$ can be seen as tensor product of states (Kronecker product) $|\iota\rangle = |\iota_1\rangle \otimes |\iota_2\rangle \otimes \ldots \otimes |\iota_n\rangle$ where $\iota_j \in \{0,1\}$. A pure state $|\psi\rangle \in \otimes^n\mathbb{C}^2$ is a superposition of the basis vectors $|\psi\rangle = \sum_{\iota=1}^{2^n} c_\iota|\iota\rangle$ with $\sum_{\iota=1}^{2^n} |c_\iota|^2 = 1$.

In the usual representation of quantum computational processes, a quantum circuit is identified with an appropriate composition of *quantum gates*, i.e. unitary operators acting on pure states of a convenient (n-fold tensor product) Hilbert space $\otimes^n \mathbb{C}^2$ [25]. Consequently, quantum gates represent time reversible evolutions of pure states of the system.

In general, a quantum system is not in a pure state. This may be caused, for example, by the non complete efficiency in the preparation procedure or by the fact that systems cannot be completely isolated from the environment, undergoing decoherence of their states. On the other hand, there are interesting processes that cannot be encoded in unitary evolutions, for example, at the end of the computation a non-unitary operation, a measurement, is applied, and the state becomes a probability distribution over pure states, or what is called a *mixed state*. In view of these facts, several authors [1, 10, 27] have paid attention to a more general model of quantum computational processes, where pure states are replaced by mixed states. This model is known as *quantum computation with mixed states*. In what follows we briefly describe the mentioned model.

Let H be a complex Hilbert space. We denote by $\mathcal{L}(H)$ the space of linear operators on H. In the model of quantum computation with mixed states, we regard a quantum state in a Hilbert space H as a *density operator* i.e., an Hermitian operator $\rho \in \mathcal{L}(H)$ that is positive semidefinite ($\rho \geq 0$) and has unit trace ($tr(\rho) = 1$). We denote by $\mathcal{D}(H)$ the set of all density operators in H. A *quantum operation* is a linear map $\mathcal{E} : \mathcal{L}(H_1) \to \mathcal{L}(H_2)$ that is trace-preserving and *completely positive*. Intuitively, completely positive means that if we embed H into some larger system, the standard lifting of $\mathcal{E}$ to the larger system preserves positive definiteness, and thus states get mapped to states. Formally, this means that for any Hilbert space K, the linear map $\mathcal{E} \otimes \mathcal{I}_K : \mathcal{L}(H_1 \otimes K) \to \mathcal{L}(H_2 \otimes K)$ where $\mathcal{I}_K$ is the identity in $\mathcal{L}(H)$, satisfies that for any $\rho \in \mathcal{L}(H_1 \otimes K)$, if $\rho > 0$ then $(\mathcal{E} \otimes \mathcal{I}_K)(\rho) > 0$. Each quantum operation $\mathcal{E}$ may be expressed as $\mathcal{E}(\rho) = \sum_i A_i \rho A_i^\dagger$ where A_i are linear operators satisfying $\sum_i A_i^\dagger A_i = I$ (Kraus representation [18]).

In the representation of quantum computational processes based on mixed states, a quantum circuit is a circuit whose inputs and outputs are labeled with density operators and whose quantum gates are labeled with quantum operations. In terms of density operators, a n-qbit $|\psi\rangle \in \otimes^n \mathbb{C}^2$ can be represented as a matrix product $\rho = |\psi\rangle\langle\psi|$, where $\langle\psi| = |\psi\rangle^\dagger$. Moreover, every unitary operator U on a Hilbert space $\otimes^m \mathbb{C}^2$ gives rise to a quantum operation $\mathcal{O}_\mathcal{U}$ such that $\mathcal{O}_\mathcal{U}(\sigma) = \mathcal{U}\sigma\mathcal{U}^\dagger$ for each $\sigma \in \mathcal{L}(H)$. In fact, quantum computation with mixed states is a generalization of the standard model based on qbits and unitary operations. We want to stress that the measurement process in quantum computation can also be described by a quantum operation, an important fact that reinforces the election of quantum operations to represent quantum gates. We refer to [1, 25, 27], for more details and motivations about quantum operations.

In this powerful model we can extend, in a natural way, the logical base of qbits and the notion of probability assigned to a qbit. In fact: we may relate to each vector of the logical basis of $\mathbb{C}^2$ one of the distinguished density operators $P_0 = |0\rangle\langle 0|$ and $P_1 = |1\rangle\langle 1|$ that represent the falsity-property and the truth-property respectively. Generalizing to the framework of n dimensions, the two special operators $P_0^{(n)} = \frac{1}{Tr(I^{n-1}\otimes P_0)} I^{n-1} \otimes P_0$ and $P_1^{(n)} = \frac{1}{Tr(I^{n-1}\otimes P_1)} I^{n-1} \otimes P_1$ (where n is even and $n \geq 2$) represent, in each space $\mathcal{D}(\otimes^n \mathbb{C}^2)$, the falsity-property and the truth-property respectively. By applying the Born rule, the probability to obtain the truth-property $P_1^{(n)}$ for a system being in the state ρ is given by the following definition:

Definition 1.2 [3], [5] Let $\rho \in \mathcal{D}(\otimes^n \mathbb{C}^2)$. Then, its *probability value* is $p(\rho) = Tr(P_1^{(n)} \rho)$.

Note that, in the particular case in which $\rho = |\psi\rangle\langle\psi|$ where $|\psi\rangle = c_0|0\rangle + c_1|1\rangle$, we obtain that $p(\rho) = |c_1|^2$. This definition of probability allows to introduce a binary relation $\leq_w$ on $\mathcal{D}(\otimes^n \mathbb{C}^2)$ in the following way:

$$\sigma \leq_w \rho \quad iff \quad p(\sigma) \leq p(\rho)$$

One can easily see that $\langle \mathcal{D}(\otimes^n \mathbb{C}^2), \leq_w \rangle$ is a preorder and it will play an important role in the rest of the paper.

2 MV-algebras and PMV-algebras

We freely use all basic notions of universal algebra that can be found in [2]. Let σ be a type of algebras and let $\mathcal{A}$ be a class of algebras of type σ. We denote by $Term_{\mathcal{A}}$ the absolutely free algebra of type σ built from the set of variables $V = \{x_1, x_2, \ldots\}$. Each element of $Term_{\mathcal{A}}$ is referred as an $\mathcal{A}$-term. For $t \in Term_{\mathcal{A}}$ we often write t as $t(x_1, x_2, \ldots, x_n)$ to indicate that the variables occurring in t are among $x_1, x_2, \ldots, x_n$. Let $A \in \mathcal{A}$. If $t(x_1, x_2, \ldots, x_n) \in Term_{\mathcal{A}}$ and $a_1, \ldots a_n \in A$, by $t^A[a_1, \ldots, a_n]$, we denote the result of the application of the term operation t^A to the elements $a_1, \ldots a_n \in A$. A *valuation* in A is a map $v : V \to A$. Of course, any valuation v in A can be uniquely extended to an $\mathcal{A}$-homomorphism $v : Term_{\mathcal{A}} \to$

A in the usual way, i.e., if $t_1, \ldots, t_n \in Term_{\mathcal{A}}$ then $v(t(t_1, \ldots, t_n)) = t^A(v(t_1), \ldots, v(t_n))$. Thus, valuations are identified with $\mathcal{A}$-homomorphisms from the absolutely free algebra. If $t, s \in Term_{\mathcal{A}}$, $A \models t = s$ means that for each valuation v in A, $v(t) = v(s)$ and $\mathcal{A} \models t = s$ means that for each $A \in \mathcal{A}$, $A \models t = s$. If $\mathcal{S}$ is a subclass of $\mathcal{A}$, $\mathcal{V}(\mathcal{S})$ denotes the variety generated by $\mathcal{S}$.

Now we introduce some basic notions in algebraic structures associated to fuzzy logic. An *MV-algebra* [4] is an algebra $\langle A, \oplus, \neg, 0\rangle$ of type $\langle 2, 2, 0\rangle$ satisfying the following equations:

MV1 $\langle A, \oplus, 0\rangle$ is an abelian monoid,

MV2 $\neg\neg x = x$,

MV3 $x \oplus \neg 0 = \neg 0$,

MV4 $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

In agreement with the usual MV-algebraic operations we define:

$$x \odot y = \neg(\neg x \oplus \neg y), \qquad x \to y = \neg x \oplus y,$$
$$x \wedge y = x \odot (x \to y), \qquad 1 = \neg 0,$$
$$x \vee y = (x \to y) \to y.$$

On each MV-algebra A we can define an order $x \leq y$ iff $x \to y = 1$. This order turns $\langle A, \wedge, \vee, 0, 1\rangle$ into a distributive bounded lattice with 1 the greatest element and 0 the smallest element.

A very important example of MV-algebra is $[0,1]_{MV} = \langle [0,1], \oplus, \neg, 0\rangle$ such that $[0,1]$ is the real unit segment and $\oplus$ and $\neg$ are defined as follows:

$$x \oplus y = \min(1, x + y) \qquad \neg x = 1 - x$$

The derivate operations in $[0,1]_{MV}$ are given by $x \odot y = \max(0, x + y - 1)$ (called *Łukasiewicz t-norm*) and $x \to y = \min(1, 1 - x + y)$. Finally the MV-lattice structure is the natural order in $[0,1]$.

A *product MV-algebra* [22, 23, 24] (for short: PMV-algebra) is an algebra $\langle A, \oplus, \bullet, \neg, 0\rangle$ of type $\langle 2, 2, 1, 0\rangle$ satisfying the following:

1 $\langle A, \oplus, \neg, 0\rangle$ is an MV-algebra,

2 $\langle A, \bullet, 1\rangle$ is an abelian monoid,

3 $x \bullet (y \odot \neg z) = (x \bullet y) \odot \neg(x \bullet z)$.

An important example of PMV-algebra is $[0,1]_{MV}$ equipped with the usual multiplication (called *product t-norm*). This algebra is denoted by $[0,1]_{PMV}$. The following are almost immediate consequences of the definition of PMV-algebras:

Lemma 2.1 *In each PMV-algebra we have*

1. $0 \bullet x = 0$,

2. *If* $a \leq b$ *then* $a \bullet x \leq b \bullet x$,

3. $x \odot y \leq x \bullet y \leq x \wedge y$.

$\square$

Proposition 2.2 [23, Lemma 2.3] *Each PMV-algebra is isomorphic to a subdirect product of linearly ordered PMV-algebras.*

$\square$

Definition 2.3 A $PMV_{\frac{1}{2^4}}$-algebra is an algebra $\langle A, \oplus, \bullet, \neg, 0, \frac{1}{2}, 1\rangle$ of type $\langle 2, 2, 1, 0, 0, 0\rangle$ satisfying the following:

1. $\langle A, \oplus, \bullet, \neg, 0, 1\rangle$ is a PMV-algebra,

2. $\neg\frac{1}{2} = \frac{1}{2}$,

3. $\frac{1}{2^4} \oplus \frac{1}{2^4} = \frac{1}{2^3}$ where $\frac{1}{2^n}$ means the term $\frac{1}{2^{n-1}} \bullet \frac{1}{2}$ $(n \geq 2)$.

It is well known that a PMV-algebra has at most a fix point of the negation [15, Lemma 2.10]. An example of $PMV_{\frac{1}{2^4}}$-algebra is $[0,1]_{PMV}$ where the fix point of the negation is $\frac{1}{2}$.

We denote $\mathcal{PMV}_{\frac{1}{2^4}}$ the variety of $PMV_{\frac{1}{2^4}}$-algebras. This variety plays a crucial role in Section 5 and Section 7.

3 The Poincaré irreversible quantum gates system

The Poincaré irreversible quantum computational system is framed in the model of quantum computation with mixed states. It takes into account a set of quantum gates –represented by quantum operations– acting on quantum mixed states –represented by density operators of $\mathcal{D}(\mathbb{C}^2)$. We first describe some basic properties of density operators in $\mathcal{D}(\mathbb{C}^2)$. Due to the fact that the Pauli matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

plus I, where I is the 2×2 identity matrix, are a basis for the set of operators over $\mathbb{C}^2$, an arbitrary density operator $\rho \in \mathcal{D}(\mathbb{C}^2)$ may be represented as

$$\rho = \frac{1}{2}(I + r_1\sigma_x + r_2\sigma_y + r_3\sigma_z)$$

where r_1, r_2, r_3 are real numbers such that $r_1^2 + r_2^2 + r_3^2 \leq 1$. When a density operator $\rho \in \mathcal{D}(\mathbb{C}^2)$ represents a pure state, it can be identified with a point (r_1, r_2, r_3) on the sphere of radius 1 (the Bloch sphere) and each $\rho \in \mathcal{D}(\mathbb{C}^2)$ that represents a mixed state with a point in the interior of the Bloch sphere. We denote this identifications as $\rho = (r_1, r_2, r_3)$. In this way $P_1 = (0, 0 - 1)$ and $P_0 = (0, 0, 1)$. An interesting feature of density operators in $\mathcal{D}(\mathbb{C}^2)$ is the following: any real number $\lambda \in [0, 1]$, uniquely determines a density operator ρ_λ given by

$$\rho_\lambda = (1 - \lambda)P_0 + \lambda P_1$$

Lemma 3.1 [6, Lemma 6.1] *Let $\rho = (r_1, r_2, r_3) \in \mathcal{D}(\mathbb{C}^2)$. Then we have:*

1. *$p(\rho) = \frac{1-r_3}{2}$.*
2. *If $\rho = \rho_\lambda$ for some $\lambda \in [0, 1]$ then $\rho = (0, 0, 1 - 2\lambda)$ and $p(\rho_\lambda) = \lambda$.*

□

Definition 3.2 The *Poincaré irreversible quantum computational system* ($\mathbb{IP}$-system) [3, 5] is given by the following set of quantum gates

- $\sigma \oplus \tau = \rho_{p(\sigma)\oplus p(\tau)}$ [Łukasiewicz gate]
- $\sigma \bullet \tau = \rho_{p(\sigma)\cdot p(\tau)}$ [IAND gate]

- $\neg\tau = \sigma_x \tau \sigma_x^\dagger$ [NOT gate]
- $\sqrt{\tau} = \begin{pmatrix} \frac{1+i}{2} & \frac{1-i}{2} \\ \frac{1-i}{2} & \frac{1+i}{2} \end{pmatrix} \tau \begin{pmatrix} \frac{1+i}{2} & \frac{1-i}{2} \\ \frac{1-i}{2} & \frac{1+i}{2} \end{pmatrix}^\dagger$ [$\sqrt{NOT}$ gate]

where σ and τ belong to $\mathcal{D}(\mathbb{C}^2)$.

By the Kraus representation mentioned in Section 1, it is immediate that $\neg$ and $\sqrt{\cdot}$ are quantum operations. We refer to [8] for a representation of the IAND gate and the Łukasiewicz gate as quantum operations. The $\mathbb{IP}$-system defines a set of operations on $\mathcal{D}(\mathbb{C}^2)$ giving rise to the structure

$$\langle \mathcal{D}(\mathbb{C}^2), \oplus, \bullet, \neg, \sqrt{\ }, P_0, \rho_{\frac{1}{2}}, P_1 \rangle$$

known as *Poincaré irreversible quantum computational algebra* (shortly IQC-algebra) [3, 5]. The following lemma gives the main properties of the IQC-algebra.

Lemma 3.3 [6, Lemma 6.1] and [7, Lemma 3.7] *Let $\tau, \sigma \in \mathcal{D}(\mathbb{C}^2)$ and let p be the probability function over $\mathcal{D}(\mathbb{C}^2)$. Then we have:*

1. $\langle \mathcal{D}(\mathbb{C}^2), \bullet \rangle$ *and* $\langle \mathcal{D}(\mathbb{C}^2), \oplus \rangle$ *are abelian monoids,*
2. $\tau \bullet P_0 = P_0$,
3. $\tau \bullet P_1 = \rho_{p(\tau)}$,
4. $p(\tau \bullet \sigma) = p(\tau)p(\sigma)$,
5. $p(\tau \oplus \sigma) = p(\tau) \oplus p(\sigma)$,
6. $\sqrt{\neg\tau} = \neg\sqrt{\tau}$,
7. $\sqrt{\sqrt{\tau}} = \neg\tau$.

Moreover if $\sigma = (r_1, r_2, r_3)$ then

8. $\neg\sigma = (r_1, -r_2, -r_3)$ *and* $\sqrt{\sigma} = (r_1, -r_3, r_2)$, *hence* $p(\neg\sigma) = \frac{1+r_3}{2}$ *and* $p(\sqrt{\sigma}) = \frac{1-r_2}{2}$,
9. $p(\sqrt{\tau \bullet \sigma}) = p(\sqrt{\tau \oplus \sigma}) = \frac{1}{2}$.

□

Taking into account Lemma 3.3-7, $\neg$ becomes a definable operation in the IQC-algebra. Recalling that in our case the assignment of probability is done via a function $p : \mathcal{D}(\mathbb{C}^2) \to [0,1]$, it is possible to establish the following equivalence relation in $\mathcal{D}(\mathbb{C}^2)$:

$$\sigma \equiv \tau \quad iff \quad p(\sigma) = p(\tau)$$

This equivalence is strongly related to the preorder $\leq_w$ mentioned at the end of Section 1. In view of Lemma 3.1 and Lemma 3.3, we can see that $\equiv$ is a $(\oplus, \bullet, \neg)$-congruence but not a $\sqrt{\cdot}$-congruence.

Proposition 3.4 *Let $[\sigma]$ be the equivalence class of $\sigma \in \mathcal{D}(\mathbb{C}^2)$ and consider the natural application $\pi_{\equiv} : \mathcal{D}(\mathbb{C}^2) \to \mathcal{D}(\mathbb{C}^2)/_{\equiv}$. Then*

1. $[\sigma] = [\sigma \bullet P_1] = [\sigma \oplus P_0] = [\rho_{p(\sigma)}]$
2. $\langle \mathcal{D}(\mathbb{C}^2)/_{\equiv}, \oplus, \bullet, \neg, [P_0], [\rho_{\frac{1}{2}}], [P_1] \rangle$ *is a $PMV_{\frac{1}{2^4}}$-algebra and the assignment* $[\rho_\lambda] \xrightarrow{f} \lambda$ *is a $\mathcal{PMV}_{\frac{1}{2^4}}$-isomorphism from $\mathcal{D}(\mathbb{C}^2)/_{\equiv}$ onto $[0,1]_{PMV}$.*
3. *The following diagram is commutative*

$$\begin{tikzcd}
\mathcal{D}(\mathbb{C}^2) \arrow[r, "p"] \arrow[d, "\pi_{\equiv}"'] & {[0,1]_{PMV}} \arrow[dl, "f", leftrightarrow] \\
\mathcal{D}(\mathbb{C}^2)/_{\equiv} &
\end{tikzcd}$$

Proof: 1) Follows from Lemma 3.3. 2) By item 1, we can consider the identification $(\mathcal{D}(\mathbb{C}^2)/_{\equiv}) = (\rho_\lambda)_{\lambda \in [0,1]}$. Then it may be easily proved that $\langle (\mathcal{D}(\mathbb{C}^2)/_{\equiv}), \oplus, \bullet, \neg, [P_0], [\rho_{\frac{1}{2}}], [P_1] \rangle$ is a $PMV_{\frac{1}{2^4}}$-algebra and $[\rho_\lambda] \xrightarrow{f} \lambda$ is a $\mathcal{PMV}_{\frac{1}{2^4}}$-isomorphism from $\mathcal{D}(\mathbb{C}^2)/_{\equiv}$ onto $[0,1]_{PMV}$. 3) Immediate.

$\square$

Remark 3.5 By Proposition 3.4-3, we can see that the assignment of probability $p : \mathcal{D}(\mathbb{C}^2) \to [0,1]$ can be identified with the natural application $\pi_{\equiv} : \mathcal{D}(\mathbb{C}^2) \to \mathcal{D}(\mathbb{C}^2)/_{\equiv}$ where $\mathcal{D}(\mathbb{C}^2)/_{\equiv}$ is endowed with a $\mathcal{PMV}_{\frac{1}{2^4}}$-structure. This crucial fact is particularly relevant in the definition of probabilistic consequence introduced in Section 6.

4 Quantum computational algebras

In this Section we introduce algebraic structures by means of simple equations in an attempt to capture the basic properties of the IQC-algebra.

The first and more basic algebraic structure associated to the $\mathbb{IP}$-system was introduced in [21] for the Łukasiewicz and the NOT gates. This is the *quasi MV-algebra* or qMV-algebra for short. A qMV-algebra is an algebra $\langle A, \oplus, \neg, 0, 1\rangle$ of type $\langle 2, 1, 0, 0\rangle$ satisfying the following equations:

Q1. $x \oplus (y \oplus z) = (x \oplus y) \oplus z$,

Q2. $\neg\neg x = x$,

Q3. $x \oplus 1 = 1$,

Q4. $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$,

Q5. $\neg(x \oplus 0) = \neg x \oplus 0$,

Q6. $(x \oplus y) \oplus 0 = x \oplus y$,

Q7. $\neg 0 = 1$.

From an intuitive point of view, a qMV-algebra can be seen as an MV-algebra which fails to satisfy the equation $x \oplus 0 = x$. We define the binary operations $\odot, \vee, \wedge, \rightarrow$ in the same way as we did for MV-algebras.

Lemma 4.1 ([21, Lemma 6]) *The following equations are satisfied in each qMV-algebra:*

1. $x \oplus y = y \oplus x$,	*5.* $x \oplus 0 = x \wedge x$,
2. $x \oplus \neg x = 1$,	*6.* $x \wedge y = y \wedge x$,
3. $x \odot \neg x = 0$,	*7.* $x \vee y = y \vee x$,
4. $0 \oplus 0 = 0$.	

□

Another algebraic structure associated to the $\mathbb{IP}$-system was introduced in [9] for the Łukasiewicz, the NOT and the $\sqrt{\text{NOT}}$ gates. These algebras are known as *square root quasi MV-algebras* or $\sqrt{qMV}$*-algebras* for short.

A $\sqrt{qMV}$-algebra is an algebra $\langle A, \oplus, \sqrt{\ }, 0, \frac{1}{2}, 1\rangle$ of type $\langle 2, 1, 0, 0, 0\rangle$ such that, upon defining $\neg x = \sqrt{\sqrt{x}}$ for all $x \in A$, the following conditions are satisfied:

SQ1. $\langle A, \oplus, \neg, 0, \frac{1}{2}, 1\rangle$ is a qMV-algebra,

SQ2. $\sqrt{\neg x} = \neg\sqrt{x}$,

SQ3. $\sqrt{x \oplus y} \oplus 0 = \sqrt{\frac{1}{2}} = \frac{1}{2}$.

In what follows we will extend the $\sqrt{qMV}$-algebra structure taking into account the basic properties of the IAND gate given in Lemma 3.3.

Definition 4.2 A $\sqrt{qPMV}$*-algebra* is an algebra $\langle A, \oplus, \bullet, \sqrt{\ }, 0, \frac{1}{2}, 1\rangle$ of type $\langle 2, 2, 1, 0, 0, 0\rangle$ satisfying the following:

1. $\langle A, \oplus, \sqrt{\ }, 0, \frac{1}{2}, 1\rangle$ is a $\sqrt{qMV}$-algebra,
2. $x \bullet y = y \bullet x$,
3. $x \bullet (y \bullet z) = (x \bullet y) \bullet z$,
4. $x \bullet 1 = x \oplus 0$,
5. $x \bullet y = (x \bullet y) \oplus 0$,
6. $x \bullet (y \odot \neg z) = (x \bullet y) \odot \neg(x \bullet z)$,
7. $\sqrt{x \bullet y} \oplus 0 = \frac{1}{2}$.

We denote by $\sqrt{q\mathcal{PMV}}$ the variety of $\sqrt{qPMV}$-algebras. It is not very hard to see that the IQC-algebra $\langle \mathcal{D}(\mathbb{C}^2), \oplus, \bullet, \neg, \sqrt{\ }, P_0, \rho_{\frac{1}{2}}, P_1\rangle$ is a $\sqrt{qPMV}$-algebra.

Let A be a $\sqrt{qPMV}$-algebra. Then we define two binary relations $\leq$ and $\equiv$ on A as follows:

$$a \leq b \quad iff \quad 1 = a \rightarrow b$$

$$a \equiv b \quad iff \quad a \leq b \ \ and \ \ b \leq a$$

Clearly, $\langle A, \leq\rangle$ is a preorder and one can also easily prove that $a \leq b$ iff $a \wedge b = a \oplus 0$ iff $a \vee b = b \oplus 0$. Moreover $a \equiv\ a \oplus 0$.

Proposition 4.3 *Let A be a $\sqrt{qPMV}$-algebra and $a, b \in A$. Then:*

1. $a \bullet 0 = 0$,

2. *If* $a \bullet b = 1$ *then* $a \oplus 0 = b \oplus 0 = 1$,

3. *If* $a \leq b$ *then* $a \bullet x \leq b \bullet x$,

4. $x \bullet y \leq x$,

5. $x \bullet (y \oplus 0) = (x \bullet y) \oplus 0$,

6. $\frac{1}{2} = \neg \frac{1}{2}$,

7. $\frac{1}{2} \oplus 0 = \frac{1}{2}$,

8. $\sqrt{x \oplus y} \oplus \sqrt{z \oplus w} = 1$.

Proof: 1) $a \bullet 0 = a \bullet (0 \odot \neg 0) = (a \bullet 0) \odot \neg (a \bullet 0) = 0$. 2) Suppose that $a \bullet b = 1$. Then $\neg(a \oplus 0) = 1 \odot \neg(a \bullet 1) = (a \bullet b) \odot \neg(a \bullet 1) = a \bullet (b \odot \neg 1) = 0$. Thus $\neg(a \oplus 0) = 0$, hence $a \oplus 0 = 1$. 3) If $a \leq b$ then $1 = a \to b = \neg(a \odot \neg b)$ and $0 = a \odot \neg b$. Using item 1. we have that $0 = x \bullet 0 = x \bullet (a \odot \neg b) = (x \bullet a) \odot \neg(x \bullet b)$. Thus, $1 = \neg((x \bullet a) \odot \neg(x \bullet b)) = (x \bullet a) \to (x \bullet b)$ resulting $(x \bullet a) \leq (x \bullet b)$. 4) Since $x \leq 1$ by item 3. we have that $x \bullet y \leq x \bullet 1 = x \oplus 0 \leq x$. 5) $x \bullet (y \oplus 0) = x \bullet (y \bullet 1) = (x \bullet y) \bullet 1 = (x \bullet y) \oplus 0$. Items 6.,7. and 8. can be easily proved.

□

Definition 4.4 Let A be a $\sqrt{qPMV}$-algebra. An element $a \in A$ is *regular* iff $a \oplus 0 = a$. We denote by $Reg(A)$ the set of regular elements.

Proposition 4.5 *Let A be a $\sqrt{qPMV}$-algebra. Then:*

1. $\langle Reg(A), \oplus, \bullet, \neg, 0, 1 \rangle$ *is a PMV-algebra.*

2. $\equiv$ *is a* $\langle \oplus, \bullet, \neg \rangle$*-congruence on A and* $\langle A/_{\equiv}, \oplus, \bullet, \neg, [0], [1] \rangle$ *is a PMV-algebra.*

3. $A/_{\equiv}$ *is $\mathcal{PMV}$-isomorphic to $Reg(A)$. This isomorphism is given by the assignment* $[x] \mapsto x \oplus 0$.

Proof: 1) From [21, Lemma 9] $\langle Reg(A), \oplus, \neg, 0, 1\rangle$ is an MV-algebra. Using Proposition 4.3-5, the operation $\bullet$ is closed in $Reg(A)$. Now from the axioms of the $\sqrt{qPMV}$-algebras, $\langle Reg(A), \oplus, \bullet, \neg, 0, 1\rangle$ results a PMV-algebra.

2) It is easy to see that $\equiv$ is a $\langle \oplus, \neg\rangle$-congruence and $\langle A/_{\equiv}, \oplus, \neg, [0]\rangle$ is an MV-algebra. For technical details see [21]. By Proposition 4.3-3, $\equiv$ is compatible with $\bullet$. Note that [1] is the identity in $\langle A/_{\equiv}, \bullet, [1]\rangle$ since $[x] \bullet [1] = [x \bullet 1] = [x \oplus 0] = [x]$. Hence, by definition of $\sqrt{qPMV}$-algebra, $\langle A/_{\equiv}, \oplus, \bullet, \neg, [0], [1]\rangle$ is a PMV-algebra.

3) Since $[x] = [x \oplus 0]$ for each $x \in A$, then φ is injective. If $x \in Reg(A)$ then $x = x \oplus 0$. Therefore $\varphi([x]) = x \oplus 0 = x$ and φ is surjective. Using Proposition 4.3-5 we have that $\varphi([x] \bullet [y]) = \varphi([x \bullet y]) = (x \bullet y) \oplus 0 = (x \oplus 0) \bullet (y \oplus 0) = \varphi([x]) \bullet \varphi([y])$. In the same way we can prove that $\varphi([x] \oplus [y]) = \varphi([x]) \oplus \varphi([y])$. By axiom Q5, $\varphi(\neg[x]) = \neg\varphi([x])$ and $\varphi([c]) = c$ for $c \in \{0, \frac{1}{2}, 1\}$ since they are regular elements in A. Thus $[x] \mapsto x \oplus 0$ is a $\mathcal{PMV}$-isomorphism.

$\square$

Definition 4.6 Let $\langle A, \oplus, \bullet, \neg, 0, 1\rangle$ be a PMV-algebra with fix point of the negation $\frac{1}{2}$. The *pair algebra* over A is the algebra

$$S_A = \langle A \times A, \oplus, \bullet, \sqrt{\ }, 0, \frac{1}{2}, 1\rangle$$

where:

$$\begin{array}{ll}
(a,b) \oplus (c,d) := (a \oplus c, \frac{1}{2}), & 0 := (0, \frac{1}{2}) \\
(a,b) \bullet (c,d) := (a \bullet c, \frac{1}{2}), & 1 := (1, \frac{1}{2}) \\
\sqrt{(a,b)} := (b, \neg a), & \frac{1}{2} := (\frac{1}{2}, \frac{1}{2})
\end{array}$$

Proposition 4.7 *Let A be a PMV-algebra with fix point of the negation $\frac{1}{2}$. Then:*

1. *The pair algebra S_A is a $\sqrt{qPMV}$-algebra.*
2. *$(a,b) \leq (c,d)$ in S_A iff $a \leq c$ in A.*
3. *$R(S_A)$ is $\mathcal{PMV}$-isomorphic to A.*

Proof: 1) It is not very hard to see that $\langle A \times A, \oplus, \sqrt{\ }, 0, \frac{1}{2}, 1\rangle$, where $\neg x = \sqrt{\sqrt{x}}$, is a $\sqrt{q MV}$-algebra. We only have to prove that S_A satisfies axioms 6 and 7 of $\sqrt{qPMV}$-algebras.

Ax 6) $x \bullet (y \odot \neg z) = (x \bullet y) \odot \neg(x \bullet z)$. In fact, $(a, b) \bullet ((c, d) \odot \neg(z, w)) = (a, b) \bullet ((c, d) \odot (\neg z, \neg w) = (a \bullet (c \odot \neg z), \frac{1}{2}) = ((a \bullet c) \odot \neg(a \bullet z), \frac{1}{2})$. On the other hand $((a, b) \bullet (c, d)) \odot \neg((a, b) \bullet (z, w)) = (a \bullet c, \frac{1}{2}) \odot (\neg(a \bullet z), \frac{1}{2}) = ((a \bullet c) \odot \neg(a \bullet z), \frac{1}{2})$.

Ax 7) $\sqrt{x \bullet y} \oplus 0 = \frac{1}{2}$. In fact: $\sqrt{(a, b) \bullet (c, d)} \oplus (0, \frac{1}{2}) = \sqrt{(a \bullet c, \frac{1}{2})} \oplus (0, \frac{1}{2}) = (\frac{1}{2}, \neg(a \bullet c)) \oplus (0, \frac{1}{2}) = (\frac{1}{2}, \frac{1}{2}) = \frac{1}{2}$.
Hence S_A is a $\sqrt{qPMV}$-algebra.

2) $(a, b) \leq (c, d)$ iff $(1, \frac{1}{2}) = (a, b) \rightarrow (c, d) = (\neg a \oplus b, \frac{1}{2})$ iff $1 = \neg a \oplus b$ in A iff $a \leq b$ in A.

3) If we consider $S_A \oplus 0 = \{(x, y) \oplus (0, \frac{1}{2}) : (x, y) \in A \times A\}$ then we have that $S_A \oplus 0 = \{(x, \frac{1}{2}) : x \in A\}$. Therefore, $S_A \oplus 0$ is $\mathcal{PMV}$-isomorphic to A. Then, by Proposition 4.5, $R(S_A)$ is $\mathcal{PMV}$-isomorphic to A.

$\square$

Definition 4.8 A $\sqrt{qPMV}$-algebra is called *flat* iff it satisfies the equation $0 = 1$.

Note that if A is flat $\sqrt{qPMV}$-algebra then $Reg(A) = \{0\}$ and $x \oplus y = x \bullet y = 0$ for each $x, y \in A$. There is a standard technique for extracting a flat $\sqrt{qPMV}$-algebra out of an arbitrary $\sqrt{qPMV}$-algebra:

Let $\langle A, \oplus, \bullet, \sqrt{\ }, 0, \frac{1}{2}, 1\rangle$ be $\sqrt{qPMV}$-algebra. Define the structure

$$Fl(A) = \langle A, \oplus, \bullet, \sqrt{\ }, 0^F, \frac{1}{2}^F, 1^F\rangle$$

where

1. $0^F = \frac{1}{2}^F = 1^F$
2. $x \oplus y = x \bullet y = 0^F$
3. $\sqrt{x} = x$

Such an algebra is easily seen to be a flat $\sqrt{qPMV}$-algebra.

Proposition 4.9 *Let A be a PMV-algebra with fix point of the negation $\frac{1}{2}$ and F be a flat $\sqrt{q}PMV$-algebra that satisfies $\sqrt{x} = x$. Consider the algebra $D = F \times S$ where S is a sub $\sqrt{q}PMV$-algebra of the pair algebra S_A. Then*

1. *$\{(0,(r_2,r_3)) : (0,(r_2,r_3)) \in D\}$ defines a sub $\sqrt{q}PMV$-algebra of D, $\sqrt{q\mathcal{PMV}}$-isomorphic to S.*

2. *The projection $\pi_S : D \to S$ is a $\sqrt{q\mathcal{PMV}}$-homomorphism.*

3. *If t contains an occurrence of $\star \in \{\oplus, \bullet\}$ then only one of the following possibilities holds: for each D-valuation v, $v(t) = (0, r, \frac{1}{2})$ or for each D-valuation v, $v(t) = (0, \frac{1}{2}, r)$.*

4. *If t contains an occurrence of $\star \in \{\oplus, \bullet\}$ then, for each D-valuation v, there exists a S-valuation v' such that $v(t) = v'(t)$.*

5. *$D \models t = s$ iff $S \models t = s$*

Proof: 1, 2) are straightforward. In what follows we identify $(x,(y,z))$ with (x,y,z) for each $(x,(y,z)) \in D$.

3) Induction on the complexity of t. Since t contains at least an occurrence of $\star \in \{\oplus, \bullet\}$, it cannot be an atomic term. Its minimum possible complexity is therefore represented by the case in which t is $t_1 \star t_2$ where each t_i is either a variable or constant. Thus for each D-valuation v, $v(t) = v(t_1 \star t_2) = (0, r, \frac{1}{2})$ for some $r \in A$. Now let our claim hold whenever the complexity of a term is less that n and t has complexity n. Suppose that t is $\sqrt{s}$. If $v(s) = (0, r, \frac{1}{2})$ then $v(\sqrt{t}) = \sqrt{(0, r, \frac{1}{2})} = (0, \frac{1}{2}, \neg r)$. If $v(s) = (0, \frac{1}{2}, r)$ then $v(\sqrt{t}) = \sqrt{(0, \frac{1}{2}, r)} = (0, r, \frac{1}{2})$. The case in which t is $s_1 \star s_2$ is immediate.

4) Let v be a D-valuation. By item 2, the composition $v' = \pi_S v$ is an S-valuation. Let t be a $\sqrt{q\mathcal{PMV}}$-term containing an occurrence of $\star \in \{\oplus, \bullet\}$. By item 3, for each D-valuation v, $v(t) = (0, r_2, r_3) = \pi_S(0, r_2, r_3) = \pi_S v(t) = v'(t)$.

5) We will consider the non trivial direction. Suppose that $D \not\models t = s$. If neither t nor s contain any occurrence of $\oplus$, then t is $\sqrt{t_1}^n$ and s is $\sqrt{s_1}^m$, where $n, m \geq 1$ are indexes of the successive application of the operation. If t_1 is a constant symbol then trivially $S \not\models t = s$ since s_1 is a variable or constant symbol. If t_1 and s_1 are both variables then $D \not\models t = s$ implies that t_1 and t_2 are different. Hence evaluate t_1 to $(0, 0, \frac{1}{2})$ and s_1 to $(0, \frac{1}{2}, \frac{1}{2})$ to get the required counterexample.

Suppose that t contains an occurrence of $\star \in \{\oplus, \bullet\}$ but s does not. We have to consider the following three sub cases:

i) If s is a constant symbol then $D \models s = s \oplus 0$ and $D \not\models t = s \oplus 0$. Let v be a D-valuation such that $v(t) \neq v(s \oplus 0)$. By item 4, there exists an S-valuation v' such that $v(t) = v'(t)$ and $v(s \oplus 0) = v'(s \oplus 0)$. Thus $v'(t) \neq v'(s \oplus 0) = v'(s)$ and $S \not\models t = s$.

ii) Suppose that s is a variable. By item 3, we first assume that $v(t)$ has the form $(0, r, \frac{1}{2})$ for each S-valuation v. Then evaluate s to $(0, 1, 0)$ to get the required counterexample. If we assume that $v(t)$ has the form $(0, \frac{1}{2}, r)$ for each D-valuation v, evaluate s to $(0, 0, 1)$ to get the required counterexample. Hence $S \not\models t = s$.

iii) With the same argument used in ii, we can prove that $S \not\models t = s$ when s is $\sqrt{s_1}^m$.

Suppose that t and s contain an occurrence of $\star \in \{\oplus, \bullet\}$. Since $D \not\models t = s$ then there exists a D-valuation v such that $v(t) \neq v(s)$. By item 4, there exists an S-valuation v' such that $v(t) = v'(t)$ and $v(s) = v'(s)$. Hence $v'(t) \neq v'(s)$ and $S \not\models t = s$.

$\square$

Proposition 4.10 *Let A be a $\sqrt{qPMV}$-algebra. Consider the $\sqrt{qPMV}$-algebra $A^* = Fl(A) \times S_{Reg(A)}$ and the application $f : A \to A^*$ such that*

$$f(x) = \begin{cases} (0, (x \oplus 0, \sqrt{x} \oplus 0)), & \textit{if } x \in Reg(A) \\ (x, (x \oplus 0, \sqrt{x} \oplus 0)), & \textit{if } x \notin Reg(A) \end{cases}$$

Then f is an injective $\sqrt{q\mathcal{PMV}}$-homomorphism.

Proof: By definition f is an injective function. We need to prove that f is a $\sqrt{q\mathcal{PMV}}$-homomorphism. In what follows we identify $(x, (y, z))$ with (x, y, z) for each $(x, (y, z)) \in A^*$.

- Let $a \in \{0, \frac{1}{2}, 1\}$. In this case $a \in Reg(A)$ and $\sqrt{a} \oplus 0 = \frac{1}{2}$. Therefore $f(a) = (0, a \oplus 0, \sqrt{a} \oplus 0) = (0, a, \frac{1}{2})$.
- Let $\star \in \{\oplus, \bullet\}$. $f(x \star y) = (0, (x \star y) \oplus 0, \sqrt{x \star y} \oplus 0) = (0, (x \star y) \oplus 0, \frac{1}{2}) = (t_x, x \oplus 0, \sqrt{x} \oplus 0) \star (t_y, y \oplus 0, \sqrt{y} \oplus 0) = f(x) \star f(y)$.
- For each $x \in A$, we define t_x as follows:

$$t_x = \begin{cases} 0, & \text{if } x \in Reg(A) \\ x, & \text{if } x \notin Reg(A) \end{cases}$$

Then $f(\sqrt{x}) = (t_{\sqrt{x}}, \sqrt{x} \oplus 0, \sqrt{\sqrt{x}} \oplus 0) = (t_{\sqrt{x}}, \sqrt{x} \oplus 0, \neg(x \oplus 0)) = \sqrt{(t_{\sqrt{x}}, x \oplus 0, \sqrt{x} \oplus 0)} = \sqrt{f(x)}$.

Thus f is a $\sqrt{q\mathcal{PMV}}$-homomorphism.

$\square$

Theorem 4.11 *Let $\mathcal{S}^{\square}$ be the sub-class of pair algebras S_A where A is a PMV-chain with fix point of the negation. Then*

$$\sqrt{q\mathcal{PMV}} = \mathcal{V}(\mathcal{S}^{\square})$$

Proof: We shall prove that $\sqrt{q\mathcal{PMV}} \models t = s$ iff $\mathcal{S}^{\square} \models t = s$. As regards the non-trivial direction, assume that $\mathcal{S}^{\square} \models t = s$. Suppose that there exists a $\sqrt{q\mathcal{PMV}}$-algebra A such that $A \not\models t = s$. By Proposition 4.10, A can be embedded in $Fl(A) \times S_{Reg(A)}$. Therefore $Fl(A) \times S_{Reg(A)} \not\models t = s$ and by Proposition 4.9-5, $S_{Reg(A)} \not\models t = s$. By Proposition 2.2 we can consider a subdirect representation $\beta : Reg(A) \hookrightarrow \Pi_{i\in I} A_i$ such that $(A_i)_{i\in I}$ is a family of PMV-chains. For each $i \in I$, let p_i be the ith-projection in A_i and consider the following function:

$$\beta_i : S_{Reg(A)} \to S_{A_i} \quad s.t. \quad (x, y) \mapsto \beta_i((x, y)) = (p_i\beta(x), p_i\beta(y))_{i\in I}$$

We shall prove that β_i is a $\sqrt{q\mathcal{PMV}}$-homomorphism.

- The preservation of $(0, \frac{1}{2})$, $(\frac{1}{2}, \frac{1}{2})$ and $(1, \frac{1}{2})$ is immediate.
- Let $\star \in \{\oplus, \bullet\}$. $\beta_i((x_1, y_1) \star (x_2, y_2)) = \beta_i(x_1 \star x_2, \frac{1}{2}) = ({x_1}_i \star {x_2}_i, \frac{1}{2}_i) = ({x_1}_i, {y_1}_i) \star ({x_2}_i, {y_2}_i) = \beta_i((x_1, y_1)) \star \beta_i((x_2, y_2))$.
- $\beta_i(\sqrt{(x, y)}) = \beta_i((y, \neg x)) = (y_i, \neg x_i) = \sqrt{(x_i, y_i)} = \sqrt{\beta_i(x, y)}$.

Thus β_i is a $\sqrt{q\mathcal{PMV}}$-homomorphism for each $i \in I$. Now we define the function

$$\beta^* : S_{Reg(A)} \to \Pi_{i\in I} S_{A_i} \quad s.t. \quad (x, y) \mapsto \beta^*((x, y)) = (\beta_i(x, y))_{i\in I}$$

Note that β^* is injective since β is a subdirect embedding. Moreover β^* is a $\sqrt{q\mathcal{PMV}}$-homomorphism since β_i is a $\sqrt{q\mathcal{PMV}}$-homomorphism for each $i \in I$. Thus $S_{Reg(A)} \not\models t = s$ implies that there exists $m \in I$ such that $S_{A_m} \not\models t = s$ which is contradiction since S_{A_m} lies in $\mathcal{S}^{\square}$. Hence $\sqrt{q\mathcal{PMV}} \models t = s$.

$\square$

We have seen that the structure of the $\sqrt{q}PMV$-algebra is a good abstraction for the IQC-algebra. However, we do not yet know whether $\sqrt{q\mathcal{PMV}} = \mathcal{V}(\mathcal{D}(\mathbb{C}^2))$. In what follows we will show that this is not the case. In order to do it, we need some preliminary results:

Lemma 4.12 *1. The set $\mathcal{D}(\mathbb{C}^2)_{y,z} = \{(0, y, z) : (0, y, z) \in \mathcal{D}(\mathbb{C}^2)\}$ defines a sub $\sqrt{q}PMV$-algebra of $\mathcal{D}(\mathbb{C}^2)$.*

2. *Consider the real interval $[-1, 1]$ equipped with the following operations: $x \oplus y = x \bullet y = 0$, $\sqrt{x} = x$ and $0^{[-1,1]} = \frac{1}{2}^{[-1,1]} = 1^{[-1,1]} = 0$. Then $F_{[-1,1]} = \langle [-1, 1], \oplus, \bullet, \sqrt{\ }, 0^{[-1,1]}, \frac{1}{2}^{[-1,1]}, 1^{[-1,1]}\rangle$ is a flat algebra.*

3. *$f : F_{[-1,1]} \times \mathcal{D}(\mathbb{C}^2)_{y,z} \to \mathcal{D}(\mathbb{C}^2)$ such that $(x,(0, y, z)) \mapsto (x, y, z)$ is a $\sqrt{q}PMV$-isomorphism.*

Proof: Straightforward.

□

Let $S_{[0,1]}$ be the pair algebra over the standard PMV-algebra $[0, 1]_{PMV}$. If we consider the set

$$D_{[0,1]} = \{(x, y) \in S_{[0,1]} : (x - \frac{1}{2})^2 + (y - \frac{1}{2})^2 \leq \frac{1}{4}\}$$

it is not very hard to see that $\langle D_{[0,1]}, \oplus, \bullet, \sqrt{\ }, 0, \frac{1}{2}, 1\rangle$ is a sub $\sqrt{q}PMV$-algebra of $S_{[0,1]}$. Moreover $Reg(D_{[0,1]}) = Reg(S_{[0,1]})$ and it is PMV-isomorphic to $[0, 1]_{PMV}$.

Proposition 4.13 *1. $\varphi : \mathcal{D}(\mathbb{C}^2)_{y,z} \to D_{[0,1]}$ such that $\varphi(y, z) = (\frac{1-z}{2}, \frac{1-y}{2})$ is a $\sqrt{q\mathcal{PMV}}$-isomorphism.*

2. *$\mathcal{D}(\mathbb{C}^2) \models t = s$ iff $D_{[0,1]} \models t = s$.*

Proof: 1) Let $\sigma = (0, b, c) \in \mathcal{D}(\mathbb{C}^2)_{y,z}$. Then $\varphi(\sigma) = (\frac{1-c}{2}, \frac{1-b}{2})$ and $(\frac{1-c}{2} - \frac{1}{2})^2 + (\frac{1-b}{2} - \frac{1}{2})^2 = \frac{1}{4}(c^2 + b^2) \leq \frac{1}{4}$. Thus the image of φ is contained in $D_{[0,1]}$. It is clear that φ is injective. Let $(a, b) \in D_{[0,1]}$. If we consider $\sigma = (0, 1 - 2b, 1 - 2a)$ then $(1 - 2b)^2 + (1 - 2a)^2 = 4(\frac{1}{2} - a)^2 + 4(\frac{1}{2} - b)^2 \leq 1$. Hence $\sigma \in \mathcal{D}(\mathbb{C}^2)_{y,z}$, $\varphi(\sigma) = (a, b)$ and φ is a surjective map. Now we prove that φ is a $\sqrt{q\mathcal{PMV}}$-homomorphism. Let $\sigma = (0, r_2, r_3)$ and $\tau = (0, s_2, s_3)$. Using Lemma 3.1 and Lemma 3.3 we have that:

- Let $\star \in \{\oplus, \bullet\}$. $\varphi(\sigma \star \tau) = \varphi(\rho_{p(\sigma)\star p(\rho)}) = \varphi(0, 0, 1 - 2(p(\sigma) \star p(\rho))) = (p(\sigma) \star p(\rho), \frac{1}{2}) = (\frac{1-r_3}{2}, \frac{1-r_2}{2}) \star (\frac{1-s_3}{2}, \frac{1-s_2}{2}) = \varphi(\sigma) \star \varphi(\tau)$.

- $\varphi(\sqrt{\sigma}) = \varphi(0, -r_3, r_2) = (\frac{1-r_2}{2}, \frac{1+r_3}{2}) = (\frac{1-r_2}{2}, 1-\frac{1-r_3}{2}) = \sqrt{(\frac{1-r_3}{2}, \frac{1-r_2}{2})} = \sqrt{\varphi(\sigma)}$.
- $\varphi(P_1) = \varphi(0,0,-1) = (1, \frac{1}{2})$, $\varphi(P_0) = \varphi(0,0,1) = (0, \frac{1}{2})$ and $\varphi(\rho_{\frac{1}{2}}) = \varphi(0,0,0) = (\frac{1}{2}, \frac{1}{2})$.

Thus φ is $\sqrt{q\mathcal{PMV}}$-isomorphism.

2) By item 1 and Lemma 4.12 we can see that $\mathcal{D}(\mathbb{C}^2)$ is $\sqrt{q\mathcal{PMV}}$-isomorphic to $F_{[-1,1]} \times D_{[0,1]}$. Hence by Proposition 4.9-5 $\mathcal{D}(\mathbb{C}^2) \models t = s$ iff $D_{[0,1]} \models t = s$.

$\square$

Proposition 4.14 *Consider the pair algebra $S_{[0,1]}$ and $a = (a_1, a_2) \in S_{[0,1]}$. Then the following conditions are equivalent:*

1. *$a \in D_{[0,1]}$,*
2. *a satisfies the equation $1 = (\frac{1}{2^4} \oplus (\frac{x \bullet x}{2^2} \oplus \frac{\sqrt{x} \bullet \sqrt{x}}{2^2})) \to (\frac{x}{2^2} \oplus \frac{\sqrt{x}}{2^2})$,*
3. *a satisfies the equation $1 = (\frac{1}{2^4} \oplus (\frac{\neg x \bullet \neg x}{2^2} \oplus \frac{\sqrt{x} \bullet \sqrt{x}}{2^2})) \to (\frac{\neg x}{2^2} \oplus \frac{\sqrt{x}}{2^2})$,*
4. *a satisfies the equation $1 = (\frac{1}{2^4} \oplus (\frac{x \bullet x}{2^2} \oplus \frac{\sqrt{\neg x} \bullet \sqrt{\neg x}}{2^2})) \to (\frac{x}{2^2} \oplus \frac{\sqrt{\neg x}}{2^2})$,*
5. *a satisfies the equation $1 = (\frac{1}{2^4} \oplus (\frac{\neg x \bullet \neg x}{2^2} \oplus \frac{\sqrt{\neg x} \bullet \sqrt{\neg x}}{2^2})) \to (\frac{\neg x}{2^2} \oplus \frac{\sqrt{\neg x}}{2^2})$,*

where $\frac{1}{2^n}$ means the term $\frac{1}{2^{n-1}} \bullet \frac{1}{2}$ and $\frac{t}{2^n}$ means the term $t \bullet \frac{1}{2^n}$ $(n \geq 2)$.

Proof: $1 \Longleftrightarrow 2$) $1 = (\frac{1}{2^4} \oplus (\frac{x \bullet x}{2^2} \oplus \frac{\sqrt{x} \bullet \sqrt{x}}{2^2})) \to (\frac{x}{2^2} \oplus \frac{\sqrt{x}}{2^2})$ is satisfied by $a = (a_1, a_2) \in S_{[0,1]}$ iff $(1, \frac{1}{2}) = ((\frac{1}{2^4}, \frac{1}{2}) \oplus ((\frac{a_1^2}{2^2}, \frac{1}{2}) \oplus (\frac{a_2^2}{2^2}, \frac{1}{2}))) \to ((\frac{a_1}{2^2}, \frac{1}{2}) \oplus (\frac{a_2}{2^2}, \frac{1}{2}))$ iff $(1, \frac{1}{2}) = (\frac{1}{2^4} \oplus \frac{a_1^2}{2^2} \oplus \frac{a_2^2}{2^2}, \frac{1}{2}) \to (\frac{a_1}{2^2} \oplus \frac{a_2}{2^2}, \frac{1}{2})$ iff $1 = (\frac{1}{2^4} \oplus \frac{a_1^2}{2^2} \oplus \frac{a_2^2}{2^2}) \to (\frac{a_1}{2^2} \oplus \frac{a_2}{2^2})$ iff $\frac{1}{2^4} \oplus \frac{a_1^2}{2^2} \oplus \frac{a_2^2}{2^2} \leq \frac{a_1}{2^2} \oplus \frac{a_2}{2^2}$ iff $\frac{1}{2^4} + \frac{a_1^2}{2^2} + \frac{a_2^2}{2^2} \leq \frac{a_1}{2^2} + \frac{a_2}{2^2}$ iff $a_1^2 - a_1 + \frac{1}{2^2} + a_2^2 - a_2 + \frac{1}{2^2} \leq \frac{1}{2^2}$ iff $(a_1 - \frac{1}{2})^2 + (a_2 - \frac{1}{2})^2 \leq \frac{1}{4}$ iff $a = (a_1, a_2) \in D_{[0,1]}$.

$1 \Longleftrightarrow 3$) $1 = (\frac{1}{2^4} \oplus (\frac{\neg x \bullet \neg x}{2^2} \oplus \frac{\sqrt{x} \bullet \sqrt{x}}{2^2})) \to (\frac{\neg x}{2^2} \oplus \frac{\sqrt{x}}{2^2})$ is satisfied by $a = (a_1, a_2) \in S_{[0,1]}$ iff $(1, \frac{1}{2}) = ((\frac{1}{2^4}, \frac{1}{2}) \oplus ((\frac{(1-a_1)^2}{2^2}, \frac{1}{2}) \oplus (\frac{a_2^2}{2^2}, \frac{1}{2}))) \to ((\frac{1-a_1}{2^2}, \frac{1}{2}) \oplus (\frac{a_2}{2^2}, \frac{1}{2}))$ iff $(1, \frac{1}{2}) = (\frac{1}{2^4} \oplus \frac{(1-a_1)^2}{2^2} \oplus \frac{a_2^2}{2^2}, \frac{1}{2}) \to (\frac{(1-a_1)}{2^2} \oplus \frac{a_2}{2^2}, \frac{1}{2})$ iff $1 = (\frac{1}{2^4} \oplus$

$\frac{(1-a_1)^2}{2^2} \oplus \frac{a_2^2}{2^2}) \to (\frac{1-a_1}{2^2} \oplus \frac{a_2}{2^2})$ iff $\frac{1}{2^4} \oplus \frac{(1-a_1)^2}{2^2} \oplus \frac{a_2^2}{2^2} \leq \frac{1-a_1}{2^2} \oplus \frac{a_2}{2^2}$ iff $\frac{1}{8} + \frac{a_1^2}{2} + \frac{a_2^2}{2} \leq \frac{a_1}{2} + \frac{a_2}{2}$ iff $(a_1 - \frac{1}{2})^2 + (a_2 - \frac{1}{2})^2 \leq \frac{1}{4}$ iff $a = (a_1, a_2) \in D_{[0,1]}$.

With the same argument we can prove $1 \iff 4)$ and $1 \iff 5)$.

□

Theorem 4.15 $\sqrt{q\mathcal{PMV}} \neq \mathcal{V}(\mathcal{D}(\mathbb{C}^2))$

Proof: By Proposition 4.13-2 and Proposition 4.14, the equation $1 = (\frac{1}{16} \oplus (\frac{x \bullet x}{4} \oplus \frac{\sqrt{x} \bullet \sqrt{x}}{4})) \to (\frac{x}{4} \oplus \frac{\sqrt{x}}{4})$ holds in $\mathcal{D}(\mathbb{C}^2)$ and fails in $S_{[0,1]}$. Hence $\sqrt{q\mathcal{PMV}} \neq \mathcal{V}(\mathcal{D}(\mathbb{C}^2))$.

□

5 Poincaré irreversible algebras

In this section we introduce and study an algebraic structure called "Poincaré irreversible algebra" with the following motivation. As shown by Proposition 4.13 the $\sqrt{q\mathcal{PMV}}$-equations which hold in the IQC-algebra coincide with the $\sqrt{q\mathcal{PMV}}$-equations which hold in $D_{[0,1]}$. Moreover, the property that characterizes $D_{[0,1]}$, i.e. a circle of radius $\frac{1}{2}$ and center $(\frac{1}{2}, \frac{1}{2})$ circumscribed in the square $[0,1] \times [0,1]$, may be captured by the set of equations given in Proposition 4.14. Since these four equations may be formulated in the language of $\sqrt{q\mathcal{PMV}}$, this allows to study a subvariety of the $\sqrt{qPMV}$-algebras that captures in a more faithful manner the basic properties of the IQC-algebra.

Definition 5.1 A *Poincaré irreversible algebra* (IP-algebra for short) is a $\sqrt{qPMV}$-algebra satisfying the following axioms:

IP1. $1 = (\frac{1}{2^4} \oplus (\frac{x \bullet x}{2^2} \oplus \frac{\sqrt{x} \bullet \sqrt{x}}{2^2})) \to (\frac{x}{2^2} \oplus \frac{\sqrt{x}}{2^2})$,

IP2. $1 = (\frac{1}{2^4} \oplus (\frac{\neg x \bullet \neg x}{2^2} \oplus \frac{\sqrt{x} \bullet \sqrt{x}}{2^2})) \to (\frac{\neg x}{2^2} \oplus \frac{\sqrt{x}}{2^2})$,

IP3. $1 = (\frac{1}{2^4} \oplus (\frac{x \bullet x}{2^2} \oplus \frac{\sqrt{\neg x} \bullet \sqrt{\neg x}}{2^2})) \to (\frac{x}{4} \oplus \frac{\sqrt{\neg x}}{2^2})$,

IP4. $1 = (\frac{1}{2^4} \oplus (\frac{\neg x \bullet \neg x}{2^2} \oplus \frac{\sqrt{\neg x} \bullet \sqrt{\neg x}}{2^2})) \to (\frac{\neg x}{2^2} \oplus \frac{\sqrt{\neg x}}{2^2})$,

IP5. $\frac{1}{2^4} \oplus \frac{1}{2^4} = \frac{1}{2^3}$.

where $\frac{1}{2^n}$ means the term $\frac{1}{2^{n-1}} \bullet \frac{1}{2}$ and $\frac{t}{2^n}$ means the term $t \bullet \frac{1}{2^n}$ $(n \geq 2)$.

Remark 5.2 In the particular case of $D_{[0,1]}$, axioms IP1,...,IP4 are all equivalent. This is due to simple arithmetic properties of the real numbers. In the general case, a $\sqrt{qPMV}$-algebra that satisfies one of these axioms does not necessarily satisfy the others. It is due to this fact that the four equations must be introduce as axioms in the definition of IP-algebra.

We denote by $\mathcal{IP}$ the subvariety of $\sqrt{q\mathcal{PMV}}$ given by the IP-algebras. Clearly $\mathcal{IP}$-homomorphisms are $\sqrt{q\mathcal{PMV}}$-homomorphisms. $D_{[0,1]}$ is an IP-algebra and constitutes the standard model for $\mathcal{IP}$. Another important examples of IP-algebras are the flat algebras.

Remark 5.3 Unfortunately we cannot give a completeness theorem for the $\mathcal{IP}$-equations of the form $t = s$ with respect to $D_{[0,1]}$. In fact, the open problem for the axiomatization of all identities in the language of $\mathcal{PMV}$ which are valid in the PMV-algebra arising from the real interval $[0, 1]$ (see [22, 16]) will appear in $\mathcal{IP}$. In view of this, we delineate a generalization of the $D_{[0,1]}$ algebra, whose role is analogous to the PMV-chains with respect to the equational theory of $\mathcal{PMV}$.

Let A be a $PMV_{\frac{1}{2^4}}$-algebra (see Definition 2.3) and S_A be the pair algebra over A. We consider the following subsets in S_A:

$$Q_1 = \{x \in S_A : 1 = (\tfrac{1}{2^4} \oplus (\tfrac{x \bullet x}{2^2} \oplus \tfrac{\sqrt{x} \bullet \sqrt{x}}{2^2})) \rightarrow (\tfrac{x}{2^2} \oplus \tfrac{\sqrt{x}}{2^2})\}$$

$$Q_2 = \{x \in S_A : 1 = (\tfrac{1}{2^4} \oplus (\tfrac{\neg x \bullet \neg x}{2^2} \oplus \tfrac{\sqrt{x} \bullet \sqrt{x}}{2^2})) \rightarrow (\tfrac{\neg x}{2^2} \oplus \tfrac{\sqrt{x}}{2^2})\}$$

$$Q_3 = \{x \in S_A : 1 = (\tfrac{1}{2^4} \oplus (\tfrac{x \bullet x}{2^2} \oplus \tfrac{\sqrt{\neg x} \bullet \sqrt{\neg x}}{2^2})) \rightarrow (\tfrac{x}{2^2} \oplus \tfrac{\sqrt{\neg x}}{2^2})\}$$

$$Q_4 = \{x \in S_A : 1 = (\tfrac{1}{2^4} \oplus (\tfrac{\neg x \bullet \neg x}{2^2} \oplus \tfrac{\sqrt{\neg x} \bullet \sqrt{\neg x}}{2^2})) \rightarrow (\tfrac{\neg x}{2^2} \oplus \tfrac{\sqrt{\neg x}}{2^2})\}$$

Then we define

$$D_A = Q1 \cap Q2 \cap Q3 \cap Q4$$

Proposition 5.4 *Let S_A be the pair algebra over the $PMV_{\frac{1}{2^4}}$-algebra A. Then:*

$$\langle D_A, \oplus, \bullet, \sqrt{\ }, \ (0, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}), (1, \frac{1}{2}) \rangle$$

is the largest IP-algebra contained in S_A as sub-$\sqrt{qPMV}$-algebra. Moreover $(a, \frac{1}{2}) \mapsto a$ defines a $\mathcal{PMV}_{\frac{1}{2^4}}$-isomorphic from $Reg(D_A)$ onto A.

Proof: We first prove that $Reg(S_A) \subseteq D_A$. Let $(a, \frac{1}{2}) \in Reg(S_A)$. We have to prove the following four cases:

Case 1: $(a, \frac{1}{2}) \in Q_1$. $\frac{1}{2^4} \oplus (\frac{x \bullet x}{2^2} \oplus \frac{\sqrt{x} \bullet \sqrt{x}}{2^2})|_{(a,\frac{1}{2})} = (\frac{1}{2^4}, \frac{1}{2}) \oplus ((\frac{1}{2^2}, \frac{1}{2}) \bullet (a, \frac{1}{2}) \bullet (a, \frac{1}{2})) \oplus ((\frac{1}{2^2}, \frac{1}{2}) \bullet \sqrt{(a, \frac{1}{2})} \bullet \sqrt{(a, \frac{1}{2})}) = (\frac{1}{2^4}, \frac{1}{2}) \oplus (\frac{a \bullet a}{2^2}, \frac{1}{2}) \oplus (\frac{1}{2^4}, \frac{1}{2}) = (\frac{a \bullet a}{2^2} \oplus \frac{1}{2^3}, \frac{1}{2})$ since A is a $PMV_{\frac{1}{2^4}}$-algebra, i.e. $\frac{1}{2^4} \oplus \frac{1}{2^4} = \frac{1}{2^3}$ holds in the first component.

$\frac{x}{2^2} \oplus \frac{\sqrt{x}}{2^2}|_{(a,\frac{1}{2})} = ((\frac{1}{2^2}, \frac{1}{2}) \bullet (a, \frac{1}{2})) \oplus ((\frac{1}{2^2}, \frac{1}{2}) \bullet \sqrt{(a, \frac{1}{2})}) = (\frac{a}{2^2} \oplus \frac{1}{2^3}, \frac{1}{2})$.
Since $\frac{a \bullet a}{2^2} \oplus \frac{1}{2^3} \leq \frac{a}{2^2} \oplus \frac{1}{2^3}$ in the $PMV_{\frac{1}{2^4}}$-algebra A, $\frac{1}{2^4} \oplus (\frac{x \bullet x}{2^2} \oplus \frac{\sqrt{x} \bullet \sqrt{x}}{2^2})|_{(a,\frac{1}{2})} \leq \frac{x}{2^2} \oplus \frac{\sqrt{x}}{2^2}|_{(a,\frac{1}{2})}$. Hence $(a, \frac{1}{2}) \in Q_1$ for each $a \in A$.

Case 2: $(a, \frac{1}{2}) \in Q_2$. $\frac{1}{2^4} \oplus (\frac{\neg x \bullet \neg x}{2^2} \oplus \frac{\sqrt{x} \bullet \sqrt{x}}{2^2})|_{(a,\frac{1}{2})} = (\frac{1}{2^4}, \frac{1}{2}) \oplus ((\frac{1}{2^2}, \frac{1}{2}) \bullet (\neg a, \frac{1}{2}) \bullet (\neg a, \frac{1}{2})) \oplus ((\frac{1}{2^2}, \frac{1}{2}) \bullet \sqrt{(a, \frac{1}{2})} \bullet \sqrt{(a, \frac{1}{2})}) = (\frac{1}{2^4}, \frac{1}{2}) \oplus (\frac{\neg a \bullet \neg a}{2^2}, \frac{1}{2}) \oplus (\frac{1}{2^4}, \frac{1}{2}) = (\frac{\neg a \bullet \neg a}{2^2} \oplus \frac{1}{2^3}, \frac{1}{2})$.

$\frac{x}{2^2} \oplus \frac{\sqrt{x}}{2^2}|_{(\neg a,\frac{1}{2})} = ((\frac{1}{2^2}, \frac{1}{2}) \bullet (\neg a, \frac{1}{2})) \oplus ((\frac{1}{2^2}, \frac{1}{2}) \bullet \sqrt{(a, \frac{1}{2})}) = (\frac{\neg a}{2^2} \oplus \frac{1}{2^3}, \frac{1}{2})$.
Since $\frac{\neg a \bullet \neg a}{2^2} \oplus \frac{1}{2^3} \leq \frac{\neg a}{2^2} \oplus \frac{1}{2^3}$ in the $PMV_{\frac{1}{2^4}}$-algebra A, $\frac{1}{2^4} \oplus (\frac{\neg x \bullet \neg x}{2^2} \oplus \frac{\sqrt{x} \bullet \sqrt{x}}{2^2})|_{(a,\frac{1}{2})} \leq \frac{\neg x}{2^2} \oplus \frac{\sqrt{x}}{2^2}|_{(a,\frac{1}{2})}$. Hence $(a, \frac{1}{2}) \in Q_2$ for each $a \in A$.

Case 3: $(a, \frac{1}{2}) \in Q_3$: $\frac{1}{2^4} \oplus (\frac{x \bullet x}{2^2} \oplus \frac{\sqrt{\neg x} \bullet \sqrt{\neg x}}{2^2})|_{(a,\frac{1}{2})} = (\frac{1}{2^4}, \frac{1}{2}) \oplus ((\frac{1}{2^2}, \frac{1}{2}) \bullet (a, \frac{1}{2}) \bullet (a, \frac{1}{2})) \oplus ((\frac{1}{2^2}, \frac{1}{2}) \bullet \sqrt{\neg(a, \frac{1}{2})} \bullet \sqrt{\neg(a, \frac{1}{2})}) = (\frac{1}{2^4}, \frac{1}{2}) \oplus (\frac{a \bullet a}{2^2}, \frac{1}{2}) \oplus (\frac{1}{2^4}, \frac{1}{2}) = (\frac{a \bullet a}{2^2} \oplus \frac{1}{2^3}, \frac{1}{2})$ since A is a $PMV_{\frac{1}{2^4}}$-algebra, i.e. $\frac{1}{2^4} \oplus \frac{1}{2^4} = \frac{1}{2^3}$ holds in the first component.

$\frac{x}{2^2} \oplus \frac{\sqrt{\neg x}}{2^2}|_{(a,\frac{1}{2})} = ((\frac{1}{2^2}, \frac{1}{2}) \bullet (a, \frac{1}{2})) \oplus ((\frac{1}{2^2}, \frac{1}{2}) \bullet \sqrt{\neg(a, \frac{1}{2})}) = (\frac{a}{2^2} \oplus \frac{1}{2^3}, \frac{1}{2})$.
Since $\frac{a \bullet a}{2^2} \oplus \frac{1}{2^3} \leq \frac{a}{2^2} \oplus \frac{1}{2^3}$ in the $PMV_{\frac{1}{2^4}}$-algebra A, $\frac{1}{2^4} \oplus (\frac{x \bullet x}{2^2} \oplus \frac{\sqrt{\neg x} \bullet \sqrt{\neg x}}{2^2})|_{(a,\frac{1}{2})} \leq \frac{x}{2^2} \oplus \frac{\sqrt{\neg x}}{2^2}|_{(a,\frac{1}{2})}$. Hence $(a, \frac{1}{2}) \in Q_3$ for each $a \in A$.

Case 4: $(a, \frac{1}{2}) \in Q_4$. $\frac{1}{2^4} \oplus (\frac{\neg x \bullet \neg x}{2^2} \oplus \frac{\sqrt{\neg x} \bullet \sqrt{\neg x}}{2^2})|_{(a,\frac{1}{2})} = (\frac{1}{2^4}, \frac{1}{2}) \oplus ((\frac{1}{2^2}, \frac{1}{2}) \bullet (\neg a, \frac{1}{2}) \bullet (\neg a, \frac{1}{2})) \oplus ((\frac{1}{2^2}, \frac{1}{2}) \bullet \sqrt{\neg(a, \frac{1}{2})} \bullet \sqrt{\neg(a, \frac{1}{2})}) = (\frac{1}{2^4}, \frac{1}{2}) \oplus (\frac{\neg a \bullet \neg a}{2^2}, \frac{1}{2}) \oplus (\frac{1}{2^4}, \frac{1}{2}) = (\frac{\neg a \bullet \neg a}{2^2} \oplus \frac{1}{2^3}, \frac{1}{2})$.

$\frac{\neg x}{2^2} \oplus \frac{\sqrt{\neg x}}{2^2}|_{(\neg a,\frac{1}{2})} = ((\frac{1}{2^2}, \frac{1}{2}) \bullet (\neg a, \frac{1}{2})) \oplus ((\frac{1}{2^2}, \frac{1}{2}) \bullet \sqrt{\neg(a, \frac{1}{2})}) = (\frac{\neg a}{2^2} \oplus \frac{1}{2^3}, \frac{1}{2})$.

Since $\frac{\neg a \bullet \neg a}{2^2} \oplus \frac{1}{2^3} \leq \frac{\neg a}{2^2} \oplus \frac{1}{2^3}$ in the $PMV_{\frac{1}{2^4}}$-algebra A, $\frac{1}{2^4} \oplus (\frac{\neg x \bullet \neg x}{2^2} \oplus \frac{\sqrt{\neg x} \bullet \sqrt{\neg x}}{2^2})|_{(a, \frac{1}{2})} \leq \frac{\neg x}{2^2} \oplus \frac{\sqrt{\neg x}}{2^2}|_{(a, \frac{1}{2})}$. Hence $(a, \frac{1}{2}) \in Q_4$ for each $a \in A$.

Hence $(a, \frac{1}{2}) \in D_A$ for each $a \in A$ and $Reg(S_A) \subseteq D_A$.

Now we prove that D_A defines a sub $\sqrt{q PMV}$-algebra of S_A.

- D_A is closed by $\star \in \{\oplus, \bullet\}$. Let $(x_1, y_1), (x_2, y_2)$ in D_A and $\star \in \{\oplus, \bullet\}$. By the precedent argument, $(x_1, y_1) \star (x_2, y_2) = (x_1 \star x_2, \frac{1}{2}) \in Reg(S_A) \subseteq D_A$. Hence, D_A is closed by $\oplus$ and $\bullet$.

- D_A is closed by $\sqrt{\cdot}$. Let $t \in D_A$. We have to prove the following four cases:

  Case 1: $\sqrt{t} \in Q_1$. $(\frac{1}{2^4} \oplus (\frac{x \bullet x}{2^2} \oplus \frac{\sqrt{x} \bullet \sqrt{x}}{2^2})) \rightarrow (\frac{x}{2^2} \oplus \frac{\sqrt{x}}{2^2})|_{\sqrt{t}} = (\frac{1}{2^4} \oplus (\frac{\sqrt{t} \bullet \sqrt{t}}{2^2} \oplus \frac{\sqrt{\sqrt{t}} \bullet \sqrt{\sqrt{t}}}{2^2})) \rightarrow (\frac{\sqrt{t}}{2^2} \oplus \frac{\sqrt{\sqrt{t}}}{2^2}) = (\frac{1}{2^4} \oplus (\frac{\sqrt{t} \bullet \sqrt{t}}{2^2} \oplus \frac{\neg t \bullet \neg t}{2^2})) \rightarrow (\frac{\sqrt{t}}{2^2} \oplus \frac{\neg t}{2^2}) = 1$ since $t \in Q_2$. Hence, $\sqrt{t} \in Q_1$.

  Case 2: $\sqrt{t} \in Q_2$. $(\frac{1}{2^4} \oplus (\frac{\neg x \bullet \neg x}{2^2} \oplus \frac{\sqrt{x} \bullet \sqrt{x}}{2^2})) \rightarrow (\frac{\neg x}{2^2} \oplus \frac{\sqrt{x}}{2^2})|_{\sqrt{t}} = (\frac{1}{2^4} \oplus (\frac{\sqrt{\neg t} \bullet \sqrt{\neg t}}{2^2} \oplus \frac{\sqrt{\sqrt{t}} \bullet \sqrt{\sqrt{t}}}{2^2})) \rightarrow (\frac{\sqrt{\neg t}}{2^2} \oplus \frac{\sqrt{\sqrt{t}}}{2^2}) = (\frac{1}{2^4} \oplus (\frac{\sqrt{\neg t} \bullet \sqrt{\neg t}}{2^2} \oplus \frac{\neg t \bullet \neg t}{2^2})) \rightarrow (\frac{\sqrt{\neg t}}{2^2} \oplus \frac{\neg t}{2^2}) = 1$ since $t \in Q_4$.

  Case 3: $\sqrt{t} \in Q_3$. $(\frac{1}{2^4} \oplus (\frac{x \bullet x}{2^2} \oplus \frac{\sqrt{\neg x} \bullet \sqrt{\neg x}}{2^2})) \rightarrow (\frac{x}{2^2} \oplus \frac{\sqrt{\neg x}}{2^2})|_{\sqrt{t}} = (\frac{1}{2^4} \oplus (\frac{\sqrt{t} \bullet \sqrt{t}}{2^2} \oplus \frac{\sqrt{\neg \sqrt{t}} \bullet \sqrt{\neg \sqrt{t}}}{2^2})) \rightarrow (\frac{\sqrt{t}}{2^2} \oplus \frac{\sqrt{\neg \sqrt{t}}}{2^2}) = (\frac{1}{2^4} \oplus (\frac{\sqrt{t} \bullet \sqrt{t}}{2^2} \oplus \frac{t \bullet t}{2^2})) \rightarrow (\frac{\sqrt{t}}{2^2} \oplus \frac{t}{2^2}) = 1$ since $t \in Q_1$. Hence, $\sqrt{t} \in Q_3$.

  Case 4: $\sqrt{t} \in Q_4$. $(\frac{1}{2^4} \oplus (\frac{\neg x \bullet \neg x}{2^2} \oplus \frac{\sqrt{\neg x} \bullet \sqrt{\neg x}}{2^2})) \rightarrow (\frac{\neg x}{2^2} \oplus \frac{\sqrt{\neg x}}{2^2})|_{\sqrt{t}} = (\frac{1}{2^4} \oplus (\frac{\sqrt{\neg t} \bullet \sqrt{\neg t}}{2^2} \oplus \frac{\sqrt{\neg \sqrt{t}} \bullet \sqrt{\neg \sqrt{t}}}{2^2})) \rightarrow (\frac{\sqrt{\neg t}}{2^2} \oplus \frac{\sqrt{\neg \sqrt{t}}}{2^2}) = (\frac{1}{2^4} \oplus (\frac{\sqrt{\neg t} \bullet \sqrt{\neg t}}{2^2} \oplus \frac{t \bullet t}{2^2})) \rightarrow (\frac{\sqrt{\neg t}}{2^2} \oplus \frac{t}{2^2}) = 1$ since $t \in Q_3$. Hence, $\sqrt{t} \in Q_4$.

  Hence $\sqrt{t} \in D_A$ and D_A is closed by $\sqrt{\cdot}$.

Thus D_A is a sub-$\sqrt{q PMV}$-algebra of S_A. Since $Reg(S_A) \subseteq D_A$, $Reg(D_A)$ is $\mathcal{PMV}_{\frac{1}{2^4}}$-isomorphic to A. By definition of D_A it is immediate that D_A is an IP-algebra. Let B be an IP-algebra such that B is a sub-$\sqrt{q PMV}$-algebra of S_A. If $a \in B$, then a satisfies IP1,..., IP4. Then $s \in D_A$ and $B \subseteq D_A$. Hence D_A is the largest IP-algebra contained in S_A as sub-$\sqrt{q PMV}$-algebra. $\square$

Theorem 5.5 *Let $\mathcal{S}^\circ$ be the sub-class of IP-algebras D_A where A is a $PMV_{\frac{1}{2^4}}$-chain. Then:*

$$\mathcal{IP} = \mathcal{V}(\mathcal{S}^\circ)$$

Proof: We shall prove that $\mathcal{IP} \models t = s$ iff $\mathcal{S}^\circ \models t = s$. As regards the non-trivial direction, assume that $\mathcal{S}^\circ \models t = s$. Suppose that there exists an IP-algebra A such that $A \not\models t = s$. By Proposition 4.10, there exists a $\sqrt{q\mathcal{PMV}}$-embedding $f : A \to Fl(A) \times S_{Reg(A)}$. Thus the image $f(A)$ is an IP-algebra and it is a sub $\sqrt{q\mathcal{PMV}}$-algebra of $Fl(A) \times S_{Reg(A)}$.

We prove that $f(A)$ is a sub IP-algebra of $Fl(A) \times D_{Reg(A)}$. Let $(a_1, a_2) \in f(A)$. Then there exists $a \in A$ such that $f(a) = (a_1, a_2)$. Since a satisfies IP1,...,IP4, $f(a)$ also satisfies these equations. It implies that a_2 satisfies IP1,...,IP4 and by Proposition 5.4, $a_2 \in D_{Reg(A)}$. This proves that $f(a) = (a_1, a_2) \in Fl(A) \times D_{Reg(A)}$ and then, $f(A)$ is a sub IP-algebra of $Fl(A) \times D_{Reg(A)}$. Consequently $Fl(A) \times D_{Reg(A)} \not\models t = s$ and, by Proposition 4.9-5, $D_{Reg(A)} \not\models t = s$.

By Proposition 2.2, consider a subdirect representation $\beta : Reg(A) \hookrightarrow \Pi_{i \in I} A_i$ such that $(A_i)_{i \in I}$ is a family of $PMV_{\frac{1}{2^4}}$-chains. For each $i \in I$, let p_i be the ith-projection in A_i and consider the following function:

$$\beta_i : D_{Reg(A)} \to S_{A_i} \quad s.t. \quad (x, y) \mapsto \beta_i((x, y)) = (p_i\beta(x), p_i\beta(y))_{i \in I}$$

Following the same argument used in the proof of Theorem 4.11 we can prove that β_i is a $\sqrt{q\mathcal{PMV}}$-homomorphism. Since $D_{Reg(A)}$ is an IP-algebra, the image $\beta_i(D_{Reg(A)})$ is an IP-algebra. Then, by Proposition 5.4, $\beta_i(D_{Reg(A)})$ is a sub IP-algebra of D_{A_i}. In other words, we can see β_i as a $\sqrt{q\mathcal{PMV}}$-homomorphism $\beta_i : D_{Reg(A)} \to D_{A_i}$ for each $i \in I$. Now we define the function

$$\beta^* : D_{Reg(A)} \to \Pi_{i \in I} D_{A_i} \quad s.t. \quad (x, y) \mapsto \beta^*((x, y)) = (\beta_i(x, y))_{i \in I}$$

β^* is injective since β is injective. Moreover β^* is a $\sqrt{q\mathcal{PMV}}$-homomorphism since β_i is a $\sqrt{q\mathcal{PMV}}$-homomorphism for each $i \in I$.

Thus $D_{Reg(A)} \not\models t = s$ implies that there exists $m \in I$ such that $D_{A_i} \not\models t = s$ which is a contradiction since $D_{A_i} \in \mathcal{S}^\circ$. Hence $\mathcal{IP} \models t = s$.

$\square$

6 Probabilistic consequence

An usual problem treated in digital techniques is the following: if T is a set of Boolean circuits and t is a Boolean circuit, we want to know if a determinate state of the outputs of the circuits of T, represented in a string of bits $0, 1$, forces a determinate state of the output of t given by a bit, either 0 or 1. As a general rule, this problem can be solved through effective procedures solving the particular case in which a set of circuits T with all outputs in state 1 forces the state 1 in the output of a circuit t.

One may naturally extend this problem by considering circuits made from assemblies of quantum gates of the $\mathbb{IP}$-system called $\mathbb{IP}$-*circuits*. In fact: let T be a set of $\mathbb{IP}$-circuits and t be an $\mathbb{IP}$-circuit. Suppose that the output of the circuits of T are labeled with density operators $(\sigma_i)_i$ such that $p(\sigma_i) = 1$ for each i. We want to know whether from the above, necessarily follows an output of t labeled with a density operator σ such that $p(\sigma) = 1$. Since each $\mathbb{IP}$-circuit can be related to a $\sqrt{q\mathcal{PMV}}$-term we can define a relation of consequence based on the preservation of probability values 1.

Definition 6.1 Let $t \in Term_{\sqrt{q\mathcal{PMV}}}$ and $T \subseteq Term_{\sqrt{q\mathcal{PMV}}}$. We say that t is a *probabilistic consequence* of T in $\mathcal{D}(\mathbb{C}^2)$ (noted $T \models^{Prob}_{\mathcal{D}(\mathbb{C}^2)} t$) iff for each valuation $e : Term_{\sqrt{q\mathcal{PMV}}} \to \mathcal{D}(\mathbb{C}^2)$, $p(e(t)) = 1$ whenever $p(e(s)) = 1$ for each $s \in T$.

Taking into account Proposition 3.4 and Remark 3.5, for each $\sqrt{q\mathcal{PMV}}$-term t and for each valuation $e : Term_{\sqrt{q\mathcal{PMV}}} \to \mathcal{D}(\mathbb{C}^2)$, the probability valued $p(e(t))$ can be identifies with $e(t \oplus 0)$. Thus we can establish the following result:

Proposition 6.2 *Let $t \in Term_{\sqrt{q\mathcal{PMV}}}$ and $T \subseteq Term_{\sqrt{q\mathcal{PMV}}}$. Then the following conditions are equivalent:*

1. $T \models^{Prob}_{\mathcal{D}(\mathbb{C}^2)} t$.
2. *For each valuation $e : Term_{\sqrt{q\mathcal{PMV}}} \to \mathcal{D}(\mathbb{C}^2)$, $e(t \oplus 0) = P_1$ whenever $e(s \oplus 0) = P_1$ for each $s \in T$.*

$\square$

The equivalence given in Proposition 6.2 allows to extend, in a natural way, the concepts of probability assignment and probabilistic consequence with respect to each IP-algebra.

Definition 6.3 Let A be an IP-algebra, $e : Term_{\sqrt{q\mathcal{PMV}}} \to A$ be a valuation and $t \in Term_{\sqrt{q\mathcal{PMV}}}$. Then we define the *generalized probability value* associated to e as $e_p(t) = e(t \oplus 0)$.

We introduce the following notation: Let $T \subseteq Term_{\sqrt{q\mathcal{PMV}}}$ and $e : Term_{\sqrt{q\mathcal{PMV}}} \rightarrow A$ be a valuation. Then $e_p(T) = 1$ means that for each $s \in T$, $e_p(s) = 1$.

Definition 6.4 Let $t \in Term_{\sqrt{q\mathcal{PMV}}}$, $T \subseteq Term_{\sqrt{q\mathcal{PMV}}}$ and A be an IP-algebra. We say that t is a *probabilistic consequence of T in A* iff for each valuation $e : Term_{\sqrt{q\mathcal{PMV}}} \rightarrow A$, if $e_p(T) = 1$ then $e_p(t) = 1$.

We preserve the notation $T \models_A^{Prob} t$ for the probabilistic consequence in A. In particular $T \models_{\mathcal{IP}}^{Prob} t$ means that $T \models_A^{Prob} t$ for each $A \in \mathcal{IP}$.

A $\sqrt{q\mathcal{PMV}}$-term t is said to be a *tautology* iff for each $A \in \mathcal{IP}$ and for each $e : Term_{\sqrt{q\mathcal{PMV}}} \rightarrow A$, $e_p(t) = 1$. Note that t is a tautology iff $\emptyset \models_{\mathcal{IP}}^{Prob} t$. Thus we use the notation $\models_{\mathcal{IP}}^{Prob} t$ in the case in which t is a tautology.

Proposition 6.5 *Let $D_A \in \mathcal{S}^\circ$ where A is a $PMV_{\frac{1}{2^4}}$-algebra. If e, e' are two valuation over D_A such that for each atomic term α, $e_p(\alpha) = e'_p(\alpha)$ and $e_p(\sqrt{\alpha}) = e'_p(\sqrt{\alpha})$ then, $e = e'$.*

Proof: By definition of valuation, we have to see that e and e' coincide over atomic terms. Let α be an atomic term. Suppose that $e(\alpha) = (a, b)$ and $e'(\alpha) = (a', b')$. On the one hand, $(a, \frac{1}{2}) = (a, b) \oplus 0 = e_p(\alpha) = e'_p(\alpha) = (a', b') \oplus 0 = (a', \frac{1}{2})$ and then, $a = a'$. On the other hand, $(b', \frac{1}{2}) = (b', \neg a') \oplus 0 = e_p(\sqrt{\alpha}) = e'_p(\sqrt{\alpha}) = (b', \neg a') \oplus 0 = (b', \frac{1}{2})$ and then, $b = b'$. Hence $e(\alpha) = e'(\alpha)$.

$\square$

7 Hilbert system for the probabilistic consequence

Let $t \in Term_{\sqrt{q\mathcal{PMV}}}$ and $T \subseteq Term_{\sqrt{q\mathcal{PMV}}}$. One may naturally consider the following decision problem: does there exist an effective procedure deciding whether $T \models_{\mathcal{IP}}^{Prob} t$? In this section we shall reformulate this problem in purely logical terms within a Hilbert-style axiomatization ($\mathcal{LIP}$) for the probabilistic consequence.

Definition 7.1 Consider the absolutely free algebra $Term_{\sqrt{q\mathcal{PMV}}}$ taking into account the following syntactic abbreviations:

$\neg t$ is a syntactic abbreviation for $\sqrt{\sqrt{t}}$,

$t_1 \odot t_2$ is a syntactic abbreviation for $\neg(\neg t_1 \oplus \neg t_2)$,

$t_1 \rightarrow t_2$ is a syntactic abbreviation for $\neg t_1 \oplus t_2$,

$t_1 \leftrightarrow t_2$ is a syntactic abbreviation for $(t_1 \rightarrow t_2) \odot (t_2 \rightarrow t_1)$,

$\frac{1}{2^n}$ is a syntactic abbreviation for $\frac{1}{2^{n-1}} \bullet \frac{1}{2}$ $(n \geq 2)$,

$\frac{t}{2^n}$ is a syntactic abbreviation for $t \bullet \frac{1}{2^n}$ $(n \geq 2)$.

An axiom of the Hilbert-style calculus $\mathcal{LIP}$ is a $\sqrt{q\mathcal{PMV}}$-term that can be written in any one of the following ways, where α, β and γ denote arbitrary terms in $Term_{\sqrt{q\mathcal{PMV}}}$:

Lukasiewicz axioms:

W1 $\alpha \rightarrow (\beta \rightarrow \alpha)$,

W2 $(\alpha \rightarrow \beta) \rightarrow ((\beta \rightarrow \gamma) \rightarrow (\alpha \rightarrow \gamma))$,

W3 $(\neg\alpha \rightarrow \neg\beta) \rightarrow (\beta \rightarrow \alpha)$,

W4 $((\alpha \rightarrow \beta) \rightarrow \beta) \rightarrow ((\beta \rightarrow \alpha) \rightarrow \alpha)$,

Constant axioms:

C1 1,

C2 $\neg 0 \leftrightarrow 1$,

C3 $\neg\boldsymbol{\frac{1}{2}} \leftrightarrow \boldsymbol{\frac{1}{2}}$,

C4 $(\frac{1}{2^4} \oplus \frac{1}{2^4}) \leftrightarrow \frac{1}{2^3}$,

Product axioms:

P1 $(\alpha \bullet \beta) \rightarrow (\beta \bullet \alpha)$,

P2 $(1 \bullet \alpha) \leftrightarrow \alpha$,

P3 $(\alpha \bullet \beta) \rightarrow \beta$,

P4 $((\alpha \bullet \beta) \bullet \gamma) \leftrightarrow (\alpha \bullet (\beta \bullet \gamma))$,

P5 $(\alpha \bullet (\beta \odot \neg\gamma)) \leftrightarrow ((\alpha \bullet \beta) \odot \neg(\alpha \bullet \gamma))$, *Sqrt axioms:*

sQ1 $\sqrt{\neg\alpha} \leftrightarrow \neg\sqrt{\alpha}$,

sQ2 $\sqrt{\alpha \star \beta} \leftrightarrow \frac{1}{2}$ where $\star \in \{\oplus, \bullet\}$,

sQ3 $\sqrt{c} \leftrightarrow \frac{1}{2}$ where $c \in \{0, \frac{1}{2}, 1\}$,

sQ4 $(\frac{1}{2^4} \oplus (\frac{\alpha \bullet \alpha}{2^2} \oplus \frac{\sqrt{\alpha}\bullet\sqrt{\alpha}}{2^2})) \rightarrow (\frac{\alpha}{2^2} \oplus \frac{\sqrt{\alpha}}{2^2})$,

sQ5 $(\frac{1}{2^4} \oplus (\frac{\neg\alpha \bullet \neg\alpha}{2^2} \oplus \frac{\sqrt{\alpha}\bullet\sqrt{\alpha}}{2^2})) \rightarrow (\frac{\neg\alpha}{2^2} \oplus \frac{\sqrt{\alpha}}{2^2})$,

sQ6 $(\frac{1}{2^4} \oplus (\frac{\alpha \bullet \alpha}{2^2} \oplus \frac{\sqrt{\neg\alpha}\bullet\sqrt{\neg\alpha}}{2^2})) \rightarrow (\frac{\alpha}{2^2} \oplus \frac{\sqrt{\neg\alpha}}{2^2})$,

sQ7 $(\frac{1}{2^4} \oplus (\frac{\neg\alpha \bullet \neg\alpha}{2^2} \oplus \frac{\sqrt{\neg\alpha}\bullet\sqrt{\neg\alpha}}{2^2})) \rightarrow (\frac{\neg\alpha}{2^2} \oplus \frac{\sqrt{\neg\alpha}}{2^2})$.

The deduction rule of $\mathcal{LIP}$ is *modus ponens*

$$\{\alpha, \alpha \rightarrow \beta\} \vdash_{\mathcal{LIP}} \beta \quad (MP)$$

Note that, axioms W1...W4, C1, C2 and MP define the same propositional system as the infinite valued Łukasiewicz calculus [4, §4]. By adding C3...C4 and P1...P5, the propositional system associated to the product Łukasiewikz logic [14, 22] with fix point of the negation [15] is obtained. sQ1, sQ2 and sQ3 axiomatize the basic properties of the operation $\sqrt{\cdot}$. Finally, sQ4...sQ7 allow to introduce in the calculus the property that characterizes the subvariety of IP-algebras with respect to the variety $\sqrt{q\mathcal{PMV}}$, as was already mentioned at the beginning of Section 5 and in Remark 5.2.

A *theory* is any set $T \subseteq Term_{\sqrt{q\mathcal{PMV}}}$. A *proof* from T is a sequence of terms $\alpha_1, ..., \alpha_n$ such that each member is either an axiom or a member of T or follows from preceding members of the sequence by modus ponens. The notation $T \vdash_{\mathcal{LIP}} \alpha$ (to be read 'α is provable form T') means that α is the last term of a proof from T. Thus the Hilbert-style calculus $\mathcal{LIP}$ is given by

$$\mathcal{LIP} = \langle Term_{\sqrt{q\mathcal{PMV}}}, \ \vdash_{\mathcal{LIP}} \rangle$$

Let T be a theory. If $T = \emptyset$ we use the notation $\vdash_{\mathcal{LIP}} \alpha$ and we say that α is a *theorem* of $\mathcal{LIP}$. T is *inconsistent* iff $T \vdash_{\mathcal{LIP}} \alpha$ for each $\alpha \in Term_{\sqrt{q\mathcal{PMV}}}$; otherwise it is *consistent.*

Lemma 7.2 *Let* $\alpha, \beta \in Term_{\sqrt{q\mathcal{PMV}}}$ *and* $T \subseteq Term_{\sqrt{q\mathcal{PMV}}}$*. Then the following items may be proved using only W1...W4, C1, C2, P1...P5 and MP.*

1. $\vdash_{\mathcal{LIP}} \alpha \to \alpha$,
2. $T \vdash_{\mathcal{LIP}} \alpha \odot \beta$ *iff* $T \vdash_{\mathcal{LIP}} \alpha$ *and* $T \vdash_{\mathcal{LIP}} \beta$,
3. $T \vdash_{\mathcal{LIP}} \alpha \leftrightarrow \beta$ *iff* $T \vdash_{\mathcal{LIP}} \alpha \to \beta$ *and* $T \vdash_{\mathcal{LIP}} \beta \to \alpha$,
4. $T \vdash_{\mathcal{LIP}} \alpha \to \beta$ *and* $T \vdash_{\mathcal{LIP}} \beta \to \gamma$ *then* $T \vdash_{\mathcal{LIP}} \alpha \to \gamma$,
5. $\vdash_{\mathcal{LIP}} \neg\neg\alpha \to \alpha$,
6. $\vdash_{\mathcal{LIP}} (\alpha \to \beta) \to (\neg\beta \to \neg\alpha)$,
7. $\vdash_{\mathcal{LIP}} (\alpha \to \beta) \to ((\alpha \oplus \gamma) \to (\beta \oplus \gamma))$,
8. $\vdash_{\mathcal{LIP}} ((\alpha \leftrightarrow \beta) \odot (\beta \leftrightarrow \gamma)) \to (\alpha \leftrightarrow \gamma)$,
9. $\vdash_{\mathcal{LIP}} (\alpha \leftrightarrow \beta) \to ((\alpha \to \gamma) \leftrightarrow (\beta \to \gamma))$,
10. $\vdash_{\mathcal{LIP}} (\alpha \leftrightarrow \beta) \to ((\gamma \to \alpha) \leftrightarrow (\gamma \to \beta))$,
11. $\vdash_{\mathcal{LIP}} (\alpha \to \beta) \to ((\gamma \bullet \alpha) \to (\gamma \bullet \beta))$.

Proof: Items 1...10 follow by observing that this result are theorems and metatheorems in the infinite valued Łukasiewicz calculus (see [13]). We prove item 11:

(1) $\vdash_{\mathcal{LIP}} \gamma \bullet (\alpha \odot \neg\beta) \to ((\alpha \odot \neg\beta))$ by Ax P3

(2) $\vdash_{\mathcal{LIP}} ((\gamma \bullet \alpha) \odot \neg(\gamma \bullet \beta)) \to \gamma \bullet (\alpha \odot \neg\beta)$ by Ax P5

(3) $\vdash_{\mathcal{LIP}} ((\gamma \bullet \alpha) \odot \neg(\gamma \bullet \beta)) \to (\alpha \odot \neg\beta)$ by 1,2, Ax W2

(4) $\vdash_{\mathcal{LIP}} (((\gamma \bullet \alpha) \odot \neg(\gamma \bullet \beta)) \to (\alpha \odot \neg\beta)) \to$
$(\neg(\alpha \odot \neg\beta) \to \neg((\gamma \bullet \alpha) \odot \neg(\gamma \bullet \beta)))$ by Ax W3

(5) $\vdash_{\mathcal{LIP}} \neg(\alpha \odot \neg\beta) \to \neg((\gamma \bullet \alpha) \odot \neg(\gamma \bullet \beta))$ by MP 3,4

(6) $\vdash_{\mathcal{LIP}} (\alpha \to \beta) \to \neg(\alpha \odot \neg\beta)$ by def $\odot$, item 1

(7) $\vdash_{\mathcal{LIP}} (\alpha \to \beta) \to \neg((\gamma \bullet \alpha) \odot \neg(\gamma \bullet \beta))$ by 5,6 , Ax W2

(8) $\vdash_{\mathcal{LIP}} \neg((\gamma \bullet \alpha) \odot \neg(\gamma \bullet \beta)) \rightarrow ((\gamma \bullet \alpha) \rightarrow (\gamma \bullet \beta))$ by def $\odot$, item 1

(9) $\vdash_{\mathcal{LIP}} (\alpha \rightarrow \beta) \rightarrow ((\gamma \bullet \alpha) \rightarrow (\gamma \bullet \beta))$ by 7,8, Ax W2

□

Proposition 7.3 *Let $\alpha, \beta \in Term_{\sqrt{q\mathcal{PMV}}}$ and $T \subseteq Term_{\sqrt{q\mathcal{PMV}}}$. Then:*

1. *Axioms of $\mathcal{LIP}$ are tautologies.*
2. *$\{\alpha, \alpha \rightarrow \beta\} \models^{Prob}_{\mathcal{IP}} \beta$.*
3. *If $T \vdash_{\mathcal{LIP}} \alpha$ then $T \models^{Prob}_{\mathcal{IP}} \alpha$.*

Proof: 1) Immediate. 2) Let $\alpha, \beta \in Term_{\sqrt{q\mathcal{PMV}}}$ and $e : \sqrt{q\mathcal{PMV}} \rightarrow A$ be an $\mathcal{IP}$-valuation such that $e_p(\alpha \rightarrow \beta) = 1$ and $e_p(\alpha) = 1$. We show that $e_p(\beta) = 1$. In fact: $1 = e_p(\alpha \rightarrow \beta) = e((\alpha \rightarrow \beta) \oplus 0) = e(\alpha \oplus 0) \rightarrow (\beta \oplus 0)$ and then $e_p(\alpha) = e(\alpha \oplus 0) \leq e(\beta \oplus 0) = e_p(\beta)$. Therefore $e_p(\alpha) = 1$ implies that $1 = e_p(\beta)$ since $1 \leq e(\beta \oplus 0) = e(\beta) \oplus 0 \in Reg(A)$ and $Reg(A)$ is a PMV-algebra (see Proposition 4.5-1). 3) Immediate from items 1 and 2.

□

Now we focus on a sub-calculus of $\mathcal{LIP}$ particularly relevant in the study of the completeness of $\mathcal{LIP}$. Consider the set $V \cup (\sqrt{x})_{x \in V}$ where V is the usual set of variables in $Term_{\sqrt{q\mathcal{PMV}}}$. Let $Term_{\sqrt{V}}$ be the smallest set $S \subseteq Term_{\sqrt{q\mathcal{PMV}}}$ such that $\{0, \frac{1}{2}, 1\} \cup V \cup (\sqrt{x})_{x \in V} \subseteq S$ and if $\alpha, \beta \in S$ then, $\neg\alpha \in S$ and $\alpha \star \beta \in S$ where $\star \in \{\oplus, \bullet\}$. Now we define the Hilbert-style calculus

$$\mathcal{L}_{\sqrt{V}} = \langle Term_{\sqrt{V}}, \vdash_{\mathcal{L}_{\sqrt{V}}} \rangle$$

given by the axioms W1 ... W4, C1 ... C4, P1 ... P5 and MP as inference rule.

Remark 7.4 By definition of $\mathcal{L}_{\sqrt{V}}$, the results of Lemma 7.2 continue to be valid in $\mathcal{L}_{\sqrt{V}}$.

Let $T \subseteq Term_{\sqrt{V}}$ i.e., a theory in $Term_{\sqrt{V}}$. Then T is said to be *complete* in $\mathcal{L}_{\sqrt{V}}$ iff for each pair of terms α, β in $Term_{\sqrt{V}}$; $T \vdash_{\sqrt{V}} \alpha \rightarrow \beta$ or $T \vdash_{\sqrt{V}} \beta \rightarrow \alpha$. Moreover T is inconsistent in $\mathcal{L}_{\sqrt{V}}$ iff $T \vdash_{\sqrt{V}} \alpha$ for each $\alpha \in Term_{\sqrt{V}}$, otherwise it is consistent in $\mathcal{L}_{\sqrt{V}}$.

Lemma 7.5 *Let T be a theory and α be a term, both in $Term_{\sqrt{V}}$. Suppose that $T \not\vdash_{\sqrt{V}} \alpha$. Then there exists a complete theory T' in $Term_{\sqrt{V}}$ such that, $T \subseteq T'$ and $T' \not\vdash_{\sqrt{V}} \alpha$.*

Proof: It follows by the same arguments used in [13, Lemma 2.4.2].

□

Let A be a $PMV_{\frac{1}{2^4}}$-algebra. $Term_{\sqrt{V}}$-*valuations* in A are functions $v : Term_{\sqrt{V}} \to A$ satisfying $v(0) = 0$, $v(\frac{1}{2}) = \frac{1}{2}$, $v(1) = 1$, $v(\neg\alpha) = \neg v(\alpha)$ and $v(\alpha \star \beta) = v(\alpha) \star v(\beta)$ where $\star \in \{\oplus, \bullet\}$. Note that for a $Term_{\sqrt{V}}$-valuation, the terms in the set $(\sqrt{x})_{x\in V}$ have no restriction in the election of the value $v(\sqrt{x})$.

Theorem 7.6 *Let T be a consistent theory in $Term_{\sqrt{V}}$. For each $\alpha \in Term_{\sqrt{V}}$ we consider the class*

$$[\alpha] = \{\beta \in Term_{\sqrt{V}} : T \vdash_{\mathcal{L}_{\sqrt{V}}} \alpha \leftrightarrow \beta\}$$

Let $L_T = \{[\alpha] : \alpha \in Term_{\sqrt{V}}\}$. If we define the following operations in L_T:

$$0 = [0] \qquad \neg[\alpha] = [\neg\alpha]$$

$$\tfrac{1}{2} = [\tfrac{1}{2}] \qquad [\alpha] * [\beta] = [\alpha * \beta] \text{ for } * \in \{\oplus, \bullet\}$$

$$1 = [1]$$

Then:

1. *$\langle L_T, \oplus, \bullet, \neg, 0, \frac{1}{2}, 1\rangle$ is a $PMV_{\frac{1}{2^4}}$-algebra.*
2. *If $\alpha \in T$ then $[\alpha] = 1$.*
3. *If T is a complete theory then L_T is a totally ordered set.*

Proof: 1) We first show that the operations are well defined on L_T. In the cases $\oplus, \neg, 0, \frac{1}{2}, 1$ we refer to [13, Lemma 2.3.12]. The case $\bullet$ follows from Lemma 7.2-11. By axioms W1 ... W4, C1 ... C4, P1 ... P5, it is not very hard to see that L_T is a $PMV_{\frac{1}{2^4}}$-algebra. 2) Follows from axiom W1. 3) Follows using the same argument as in [13, Lemma 2.4.2].

□

We will refer to L_T as the *Lindenbaum algebra* associated to the theory $T \subseteq Term_{\sqrt{V}}$. Clearly, the natural application $\alpha \mapsto [\alpha]$ is a $Term_{\sqrt{V}}$-valuation in L_T.

Definition 7.7 We define the $\sqrt{V}$-*translation* $\alpha \xrightarrow{t} \alpha_t$ as the application $t : Term_{\sqrt{q\mathcal{PMV}}} \to Term_{\sqrt{V}}$ such that:

$$x \overset{t}{\mapsto} x \text{ and } \sqrt{x} \overset{t}{\mapsto} \sqrt{x} \quad \text{for each } x \in V,$$

$$c \overset{t}{\mapsto} c \text{ and } \sqrt{c} \overset{t}{\mapsto} \tfrac{1}{2} \quad \text{for each } c \in \{0, \tfrac{1}{2}, 1\},$$

$$\neg\alpha \overset{t}{\mapsto} \neg(\alpha_t),$$

$$\sqrt{\neg\alpha} \overset{t}{\mapsto} (\neg\sqrt{\alpha})_t,$$

$$\sqrt{\alpha \star \beta} \overset{t}{\mapsto} \tfrac{1}{2} \quad \text{for each binary connective } \star,$$

$$\alpha \star \beta \overset{t}{\mapsto} \alpha_t \star \beta_t \quad \text{for each binary connective } \star$$

If T is a theory in $Term_{\sqrt{q\mathcal{PMV}}}$, then we define the $\sqrt{V}$-translation over T as the set $T_t = \{\alpha_t : \alpha \in T\}$.

Proposition 7.8 *Let $\alpha \in Term_{\sqrt{q\mathcal{PMV}}}$. Then:*

$$\vdash_{\mathcal{LIP}} \alpha \leftrightarrow \alpha_t$$

Proof: We use induction on the complexity of terms. Let α be an atomic term. By definition of $\sqrt{V}$-translation, Lemma 7.2-1 and axiom sQ3 of $\mathcal{LIP}$ it is clear that $\vdash_{\mathcal{LIP}} \alpha \leftrightarrow \alpha_t$ and $\vdash_{\mathcal{LIP}} \sqrt{\alpha} \leftrightarrow (\sqrt{\alpha})_t$.

Suppose that $\vdash_{\mathcal{LIP}} \alpha \leftrightarrow \alpha_t$ and $\vdash_{\mathcal{LIP}} \beta \leftrightarrow \beta_t$.

- By Lemma 7.2-6 we have that $\vdash_{\mathcal{LIP}} \neg\alpha \leftrightarrow \neg\alpha_t$.
- Let $\star \in \{\oplus, \bullet\}$. Then we have that:

(1)	$\vdash_{\mathcal{LIP}} \alpha \to \alpha_t$	
(2)	$\vdash_{\mathcal{LIP}} (\alpha \to \alpha_t) \to ((\alpha \star \beta) \to (\alpha_t \star \beta))$	by Lemma 7.2, item 7 or 11
(3)	$\vdash_{\mathcal{LIP}} (\alpha \star \beta) \to (\alpha_t \star \beta)$	MP 1-2
(4)	$\vdash_{\mathcal{LIP}} \beta \to \beta_t$	
(5)	$\vdash_{\mathcal{LIP}} (\beta \to \beta_t) \to ((\alpha_t \star \beta) \to (\alpha_t \star \beta_t))$	by Lemma 7.2, item 7 or 11
(6)	$\vdash_{\mathcal{LIP}} (\alpha_t \star \beta) \to (\alpha_t \star \beta_t)$	MP 4-5
(7)	$\vdash_{\mathcal{LIP}} (\alpha \star \beta) \to (\alpha_t \star \beta_t)$	by Lemma 7.2-4

By the same argument we can prove that $\vdash_{\mathcal{LIP}} (\alpha_t \star \beta_t) \to (\alpha \star \beta)$. Hence $\vdash_{\mathcal{LIP}} (\alpha \star \beta) \leftrightarrow (\alpha \star \beta)_t$.

- If α is $\sqrt{\gamma}$ then we have to consider two cases:

  i) γ is $\gamma_1 \star \gamma_2$ such that $\star \in \{\oplus, \bullet\}$. Then $\alpha_t = (\sqrt{\gamma})_t = (\sqrt{\gamma_1 \star \gamma_2})_t = \frac{1}{2}$. By Axiom sQ2, $\vdash_{\mathcal{LIP}} (\sqrt{\gamma}) \leftrightarrow \frac{1}{2}$. Hence $\vdash_{\mathcal{LIP}} \alpha \leftrightarrow \alpha_t$.

  ii) γ is $\sqrt{\gamma_1}$. Then $\alpha_t = (\sqrt{\sqrt{\gamma_1}})_t = (\neg\gamma_1)_t = \neg(\gamma_1)_t$. By inductive hypothesis $\vdash_{\mathcal{LIP}} \gamma_1 \leftrightarrow (\gamma_1)_t$ and then $\vdash_{\mathcal{LIP}} \neg\gamma_1 \leftrightarrow \neg(\gamma_1)_t$. Hence $\vdash_{\mathcal{LIP}} \sqrt{\sqrt{\gamma_1}} \leftrightarrow \neg\gamma_1$ and $\vdash_{\mathcal{LIP}} \alpha \leftrightarrow \alpha_t$.

□

Taking into account the axiom sQ4,... sQ7, we introduce, in the following definition, the theory $T_D \subseteq Term_{\sqrt{V}}$ which will allow to establish a relation between proofs in $\mathcal{LIP}$ and proofs in $\mathcal{L}_{\sqrt{V}}$.

Definition 7.9 We consider the following sets of terms in $Term_{\sqrt{V}}$

$$T_1 = \{(\tfrac{1}{2^4} \oplus (\tfrac{s \bullet s}{2^2} \oplus \tfrac{\sqrt{s}\bullet\sqrt{s}}{2^2})) \rightarrow (\tfrac{s}{2^2} \oplus \tfrac{\sqrt{s}}{2^2}) : s \in V \cup \{0, \tfrac{1}{2}, 1\}\},$$

$$T_2 = \{(\tfrac{1}{2^4} \oplus (\tfrac{\neg s \bullet \neg s}{2^2} \oplus \tfrac{\sqrt{s}\bullet\sqrt{s}}{2^2})) \rightarrow (\tfrac{\neg s}{2^2} \oplus \tfrac{\sqrt{s}}{2^2}) : s \in V \cup \{0, \tfrac{1}{2}, 1\}\},$$

$$T_3 = \{(\tfrac{1}{2^4} \oplus (\tfrac{s \bullet s}{2^2} \oplus \tfrac{\sqrt{\neg s}\bullet\sqrt{\neg s}}{2^2})) \rightarrow (\tfrac{s}{2^2} \oplus \tfrac{\sqrt{\neg s}}{2^2}) : s \in V \cup \{0, \tfrac{1}{2}, 1\}\},$$

$$T_4 = \{(\tfrac{1}{2^4} \oplus (\tfrac{\neg s \bullet \neg s}{2^2} \oplus \tfrac{\sqrt{\neg s}\bullet\sqrt{\neg s}}{2^2})) \rightarrow (\tfrac{\neg s}{2^2} \oplus \tfrac{\sqrt{\neg s}}{2^2}) : s \in V \cup \{0, \tfrac{1}{2}, 1\}.$$

Then we define:

$$T_D = T_1 \cup T_2 \cup T_3 \cup T_4$$

Proposition 7.10 *Let $\alpha \in Term_{\sqrt{q\mathcal{PMV}}}$. Then:*

1. $T_D \vdash_{\mathcal{L}_{\sqrt{V}}} ((\frac{1}{2^4} \oplus (\frac{\alpha\bullet\alpha}{2^2} \oplus \frac{\sqrt{\alpha}\bullet\sqrt{\alpha}}{2^2})) \rightarrow (\frac{\alpha}{2^2} \oplus \frac{\sqrt{\alpha}}{2^2}))_t$.
2. $T_D \vdash_{\mathcal{L}_{\sqrt{V}}} ((\frac{1}{2^4} \oplus (\frac{\neg\alpha\bullet\neg\alpha}{2^2} \oplus \frac{\sqrt{\alpha}\bullet\sqrt{\alpha}}{2^2})) \rightarrow (\frac{\neg\alpha}{2^2} \oplus \frac{\sqrt{\alpha}}{2^2}))_t$.
3. $T_D \vdash_{\mathcal{L}_{\sqrt{V}}} ((\frac{1}{2^4} \oplus (\frac{\alpha\bullet\alpha}{2^2} \oplus \frac{\sqrt{\neg\alpha}\bullet\sqrt{\neg\alpha}}{2^2})) \rightarrow (\frac{\alpha}{2^2} \oplus \frac{\sqrt{\neg\alpha}}{2^2}))_t$.
4. $T_D \vdash_{\mathcal{L}_{\sqrt{V}}} ((\frac{1}{2^4} \oplus (\frac{\neg\alpha\bullet\neg\alpha}{2^2} \oplus \frac{\sqrt{\neg\alpha}\bullet\sqrt{\neg\alpha}}{2^2})) \rightarrow (\frac{\neg\alpha}{2^2} \oplus \frac{\sqrt{\neg\alpha}}{2^2}))_t$.

Proof: Let $\alpha \in Term_{\sqrt{q\mathcal{PMV}}}$. For the sake of simplicity we use the following notation:

α^1_t means $((\frac{1}{2^4} \oplus (\frac{\alpha\bullet\alpha}{2^2} \oplus \frac{\sqrt{\alpha}\bullet\sqrt{\alpha}}{2^2})) \rightarrow (\frac{\alpha}{2^2} \oplus \frac{\sqrt{\alpha}}{2^2}))_t$,

α_t^2 means $((\frac{1}{2^4} \oplus (\frac{\neg\alpha \bullet \neg\alpha}{2^2} \oplus \frac{\sqrt{\alpha} \bullet \sqrt{\alpha}}{2^2})) \rightarrow (\frac{\neg\alpha}{2^2} \oplus \frac{\sqrt{\alpha}}{2^2}))_t$,
α_t^3 means $((\frac{1}{2^4} \oplus (\frac{\alpha \bullet \alpha}{2^2} \oplus \frac{\sqrt{\neg\alpha} \bullet \sqrt{\neg\alpha}}{2^2})) \rightarrow (\frac{\alpha}{2^2} \oplus \frac{\sqrt{\neg\alpha}}{2^2}))_t$,
α_t^4 means $((\frac{1}{2^4} \oplus (\frac{\neg\alpha \bullet \neg\alpha}{2^2} \oplus \frac{\sqrt{\neg\alpha} \bullet \sqrt{\neg\alpha}}{2^2})) \rightarrow (\frac{\neg\alpha}{2^2} \oplus \frac{\sqrt{\neg\alpha}}{2^2}))_t$.

We use induction on the complexity of α.

- The case $\alpha \in V \cup \{0, \frac{1}{2}, 1\}$ is immediate from definition of T_D.
- Suppose that α is $\alpha_1 \star \alpha_2$ where $\star \in \{\oplus, \bullet\}$.

By Axiom P3, Axiom C4 and Lemma 7.2-4 it follows that for each $\alpha \in Term_{\sqrt{V}}$

$$\vdash_{\mathcal{L}_{\sqrt{V}}} (\frac{1}{2^4} \oplus (\frac{\alpha \bullet \alpha}{2^2} \oplus \frac{1}{2^4})) \rightarrow (\frac{\alpha}{2^2} \oplus \frac{1}{2^3}) \tag{1}$$

We prove that $T_D \vdash_{\mathcal{L}_{\sqrt{V}}} \alpha_t^1$. By definition of $\sqrt{V}$-translation we have that $\alpha_t^1 = ((\frac{1}{2^4} \oplus (\frac{(\alpha_1 \star \alpha_2) \bullet (\alpha_1 \star \alpha_2)}{2^2} \oplus \frac{\sqrt{\alpha_1 \star \alpha_2} \bullet \sqrt{\alpha_1 \star \alpha_2}}{2^2})) \rightarrow (\frac{\alpha_1 \star \alpha_2}{2^2} \oplus \frac{\sqrt{\alpha_1 \star \alpha_2}}{2^2}))_t = (\frac{1}{2^4} \oplus (\frac{(\alpha_1 \star \alpha_2)_t \bullet (\alpha_1 \star \alpha_2)_t}{2^2} \oplus \frac{1}{2^4}) \rightarrow ((\frac{(\alpha_1 \star \alpha_2)_t}{2^2} \oplus \frac{1}{2^3})$. Since $(\alpha_1 \star \alpha_2)_t \in Term_{\sqrt{V}}$, by (1), we have $\vdash_{\mathcal{L}_{\sqrt{V}}} \alpha_t^1$ and $T_D \vdash_{\mathcal{L}_{\sqrt{V}}} \alpha_t^1$. Cases $T_D \vdash_{\mathcal{L}_{\sqrt{V}}} \alpha_t^2$, $T_D \vdash_{\mathcal{L}_{\sqrt{V}}} \alpha_t^3$ and $T_D \vdash_{\mathcal{L}_{\sqrt{V}}} \alpha_t^4$ follow in a similar way.

- Suppose α is $\sqrt{\beta}$.

We prove that $T_D \vdash_{\mathcal{L}_{\sqrt{V}}} \alpha_t^1$. By definition of $\sqrt{V}$-translation we have that $\alpha_t^1 = ((\frac{1}{2^4} \oplus (\frac{\sqrt{\beta} \bullet \sqrt{\beta}}{2^2} \oplus \frac{\sqrt{\sqrt{\beta}} \bullet \sqrt{\sqrt{\beta}}}{2^2})) \rightarrow (\frac{\sqrt{\beta}}{2^2} \oplus \frac{\sqrt{\sqrt{\beta}}}{2^2}))_t = ((\frac{1}{2^4} \oplus (\frac{(\sqrt{\beta})_t \bullet (\sqrt{\beta})_t}{2^2} \oplus \frac{\neg\beta_t \bullet \neg\beta_t}{2^2})) \rightarrow (\frac{(\sqrt{\beta})_t}{2^2} \oplus \frac{\neg\beta_t}{2^2})$. By inductive hypothesis we have $T_D \vdash_{\mathcal{L}_{\sqrt{V}}} \beta_t^2$ and by Lemma 7.2 it is straightforward to see that

$$T_D \vdash_{\mathcal{L}_{\sqrt{V}}} \beta_t^2 \leftrightarrow ((\frac{1}{2^4} \oplus (\frac{(\sqrt{\beta})_t \bullet (\sqrt{\beta})_t}{2^2} \oplus \frac{\neg\beta_t \bullet \neg\beta_t}{2^2})) \rightarrow (\frac{(\sqrt{\beta})_t}{2^2} \oplus \frac{\neg\beta_t}{2^2}))$$

Hence, by Lemma 7.2-3, $T_D \vdash_{\mathcal{L}_{\sqrt{V}}} \alpha_t^1$. Cases $T_D \vdash_{\mathcal{L}_{\sqrt{V}}} \alpha_t^2$, $T_D \vdash_{\mathcal{L}_{\sqrt{V}}} \alpha_t^3$ and $T_D \vdash_{\mathcal{L}_{\sqrt{V}}} \alpha_t^4$ follow in a similar way.

$\square$

Theorem 7.11 *Let $\alpha \in Term_{\sqrt{q\mathcal{PMV}}}$ and $T \subseteq Term_{\sqrt{q\mathcal{PMV}}}$. Then:*

$$T \vdash_{\mathcal{LIP}} \alpha \quad iff \quad T_t \cup T_D \vdash_{\mathcal{L}_{\sqrt{V}}} \alpha_t$$

Proof: Suppose that $T \vdash_{\mathcal{LIP}} \alpha$. We use induction on the length of the proof of α noted by $Length(\alpha)$. If $Length(\alpha) = 1$ then we have the following possibilities:

1. α is one of the axioms W1...W4, C1...C4, P1...P5. In this case α_t results an axiom of $\mathcal{L}_{\sqrt{V}}$ and $\vdash_{\mathcal{L}_{\sqrt{V}}} \alpha_t$.
2. α is one of the axioms sQ1...sQ3. In this case α_t looks like $\beta \leftrightarrow \beta$ in $Term_{\sqrt{V}}$. Then, by Proposition 7.2-1, $\vdash_{\mathcal{L}_{\sqrt{V}}} \alpha_t$.
3. α is one of the axioms sQ4...sQ7. In this case, by Proposition 7.10, $T_D \vdash_{\mathcal{L}_{\sqrt{V}}} \alpha_t$.
4. If $\alpha \in T$ then $\alpha_t \in T_t$. Hence, $T_t \vdash_{\mathcal{L}_{\sqrt{V}}} \alpha_t$.

Suppose that the theorem is valid for $Length(\alpha) < n$. We consider $Lengh(\alpha) = n$. Thus we have a proof of α from T as follows

$$\alpha_1, \cdots, \alpha_m \to \alpha, \cdots, \alpha_m, \cdots, \alpha_{n-1}, \alpha$$

obtaining α by MP from $\alpha_m \to \alpha$ and α_m. Using the inductive hypothesis we have that $T_t \cup T_D \vdash_{\mathcal{L}_{\sqrt{V}}} (\alpha_m \to \alpha)_t$ and $T_t \cup T_D \vdash_{\mathcal{L}_{\sqrt{V}}} (\alpha_m)_t$. Taking into account that $(\alpha_m \to \alpha)_t = (\alpha_m)_t \to \alpha_t$, by MP, we have $T_t \cup T_D \vdash_{\mathcal{L}_{\sqrt{V}}} \alpha_t$.

For the converse, suppose that $T_t \cup T_D \vdash_{\mathcal{L}_{\sqrt{V}}} \alpha_t$. Then there exist two subsets $\{\beta_1, \cdots, \beta_n\} \subseteq T$ and $\{\gamma_1, \cdots, \gamma_m\} \subseteq T_D$ such that

$$\{(\beta_1)_t, \cdots, (\beta_n)_t, \gamma_1, \cdots, \gamma_m\} \vdash_{\mathcal{L}_{\sqrt{V}}} \alpha_t$$

Consequently $\{(\beta_1)_t, \cdots, (\beta_n)_t, \gamma_1, \cdots, \gamma_m\} \vdash_{\mathcal{LIP}} \alpha_t$. By Lemma 7.8 we have that $\vdash_{\mathcal{LIP}} \alpha \leftrightarrow \alpha_t$ and $\vdash_{\mathcal{LIP}} \beta_i \leftrightarrow (\beta_i)_t$ for each $i \in \{1, \cdots, n\}$. Moreover, by Axiom sQ4...sQ7, $\vdash_{\mathcal{LIP}} \gamma_j$ for each $j \in \{1, \cdots, m\}$. Thus $\{\beta_1, \cdots, \beta_n\} \vdash_{\mathcal{LIP}} \alpha$ and $T \vdash_{\mathcal{LIP}} \alpha$.

$\square$

Corollary 7.12 *Let $\alpha \in Term_{\sqrt{q\mathcal{PMV}}}$. Then, $\vdash_{\mathcal{LIP}} \alpha$ iff $T_D \vdash_{\mathcal{L}_{\sqrt{V}}} \alpha_t$.*

$\square$

Let S_A be the pair algebra over the $\mathcal{PMV}_{\frac{1}{2^4}}$-chain A. Consider the sub-algebra D_A of S_A (i.e. the IP-algebra defined in Proposition 5.5). We introduce the following sets:

$$E_{D_A} = \{\text{valuations } e : Term_{\sqrt{q\mathcal{PMV}}} \to D_A\}$$

$$V_{D_A} = \{Term_{\sqrt{V}}\text{-valuations } v : Term_{\sqrt{V}} \to Reg(D_A) \ s.t.\ v(T_D) = 1\}$$

Proposition 7.13 *Let $e \in E_{D_A}$ and the restriction $v_e = e_p \mid_{Term_{\sqrt{q\mathcal{PMV}}}}$ where $e_p(t)$ is the generalized probability value. Then the assignment $e \mapsto v_e$ is a bijection $E_{D_A} \to V_D$ such that $e_p(\alpha) = v_e(\alpha_t)$.*

Proof: We first prove that $e \mapsto v_e$ is well defined in the sense that $v_e \in V_D$. Let $\alpha \in T_D$. Then $v_e(\alpha) = e_p(\alpha) = e(\alpha \oplus 0) = e(\alpha) = 1$ since $D_A \in \mathcal{IP}$. Hence $v_e(T_D) = 1$.

We prove the injectivity. Suppose that $v_{e_1} = v_{e_2}$. Let t be an atomic term in $term_{\mathcal{IP}}$. Then we have that $e_{1p}(t) = v_{e_1}(t) = v_{e_2}(t) = e_{2p}(t)$ and $e_{1p}(\sqrt{t}) = v_{e_1}(\sqrt{t}) = v_{e_2}(\sqrt{t}) = e_{2p}(\sqrt{t})$. Therefore by Proposition 6.5, $e_1 = e_2$ and $e \mapsto v_e$ is injective.

Now we prove the surjectivity. Let $v \in V_{D_A}$. By Proposition 5.4, consider the $\mathcal{PMV}_{\frac{1}{2^4}}$-isomorphism $g : Reg(D_A) \to A$ given by $g(a, \frac{1}{2}) = a$ and define the valuation $e : Term_{\sqrt{q\mathcal{PMV}}} \to D_A$ such that for each atomic term t in $Term_{\sqrt{q\mathcal{PMV}}}$ $e(t) = (gv(t), gv(\sqrt{t}))$. By induction on the complexity of terms we prove that $v_e = v$.

Let t be an atomic term in $Term_{\sqrt{V}}$.

- The case $t \in \{0, \frac{1}{2}, 1\}$ is immediate.
- If t is a variable x then, $v_e(x) = e_p(x) = e(x \oplus 0) = e(x) \oplus 0 = (gv(x), gv(\sqrt{x})) \oplus 0 = (gv(x), \frac{1}{2}) = v(x)$.
- If t is $\sqrt{x}$ where x is variable then, $v_e(\sqrt{x}) = e_p(\sqrt{x}) = e(\sqrt{x} \oplus 0) = e(\sqrt{x}) \oplus 0 = \sqrt{e(x)} \oplus 0 = \sqrt{(gv(x), gv(\sqrt{x}))} \oplus 0 = (gv(\sqrt{x}), \neg gv(x)) \oplus 0 = (gv(\sqrt{x}), \frac{1}{2}) = v(\sqrt{x})$.

That constitutes the base of the induction in the language $Term_{\sqrt{V}}$.

Now let our claim hold whenever the complexity of $Term_{\sqrt{V}}$-terms is less than n and α has complexity n.

- if $\alpha \in Term_{\sqrt{V}}$ is $\alpha_1 \star \alpha_2$ where $\star \in \{\oplus, \bullet\}$ then $e_p(\alpha) = e(\alpha \oplus 0) = e((\alpha_1 \star \alpha_2) \oplus 0) = e((\alpha_1 \oplus 0) \star (\alpha_2 \oplus 0)) = e(\alpha_1 \oplus 0) \star e(\alpha_2 \oplus 0) = e_p(\alpha_1) \star e_p(\alpha_2) = v(\alpha_1) \star v(\alpha_2) = v(\alpha_1 \star \alpha_2)$.

- if $\alpha \in Term_{\sqrt{V}}$ is $\neg\alpha_1$ then, $v_e(\alpha) = e_p(\neg\alpha_1) = e(\neg\alpha_1 \oplus 0) = e(\neg(\alpha_1 \oplus 0)) = \neg e(\alpha_1 \oplus 0) = \neg e_p(\alpha_1) = \neg v(\alpha_1) = v(\neg\alpha_1) = v(\alpha)$.

Thus $v = v_e$ and $e \mapsto v_e$ is a bijection from E_{D_A} onto V_{D_A}.

Let $e \in E_{D_A}$. By induction on the complexity of terms we prove that for each $\alpha \in Term_{\sqrt{q\mathcal{PMV}}}$, $e_p(\alpha) = v_e(\alpha_t)$.

If α is an atomic term then $e_p(\alpha) = e_p(\alpha_t) = v_e(\alpha_t)$. Now let our claim hold whenever the complexity of the term is less than n and α have complexity n.

Suppose that α is $\alpha_1 \star \alpha_2$ where $\star \in \{\oplus, \bullet\}$. Then $e_p(\alpha) = e_p(\alpha_1 \star \alpha_2) = e((\alpha_1 \star \alpha_2) \oplus 0) = e((\alpha_1 \oplus 0) \star (\alpha_2 \oplus 0)) = e(\alpha_1 \oplus 0) \star e(\alpha_2 \oplus 0) = e_p(\alpha_1) \star e_p(\alpha_2) = v_e(\alpha_{1t}) \star v_e(\alpha_{2t}) = v_e(\alpha_{1t} \star \alpha_{2t}) = v_e(\alpha_t)$.

Suppose that α is $\sqrt{\alpha_1}$. Let us consider the following cases:

- α_1 is an atomic term. Then its follows from the fact that $(\sqrt{\alpha_1})_t = \sqrt{\alpha_1}$.
- α is $\sqrt{\sqrt{\alpha_1}}$. Then $e_p(\alpha) = e_p(\sqrt{\sqrt{\alpha_1}}) = \neg e_p(\alpha_1) = \neg v_e(\alpha_{1t}) = v_e(\neg\alpha_{1t}) = v_e((\sqrt{\sqrt{\alpha_1}})_t) = v_e(\alpha)$.
- α_1 is $\sqrt{\alpha_2 \star \alpha_3}$ where $\star \in \{\oplus, \bullet\}$. Then $e_p(\alpha) = e_p(\sqrt{\alpha_2 \star \alpha_3}) = (\frac{1}{2}, \frac{1}{2}) = e_p(\frac{1}{2}) = v_e((\sqrt{\alpha_2 \star \alpha_3})_t) = v_e(\alpha_t)$.

Hence $e_p(\alpha) = v_e(\alpha_t)$ for each $\alpha \in Term_{\sqrt{q\mathcal{PMV}}}$.

□

Theorem 7.14 *Let T be a theory and α be a term both in $Term_{\sqrt{q\mathcal{PMV}}}$ then*

$$T \models^{Prob}_{\mathcal{IP}} \alpha \quad iff \quad T \vdash_{\mathcal{LIP}} \alpha$$

Proof: We assume that T is consistent. Suppose that $T \models^{Prob}_{\mathcal{IP}} \alpha$ but $T \not\vdash_{\mathcal{LIP}} \alpha$. Then, by Theorem 7.11, $T_t \cup T_D \not\vdash_{\mathcal{L}_{\sqrt{V}}} \alpha_t$. By Lemma 7.5 and Theorem 7.6, there exists a complete theory $T' \subseteq Term_{\sqrt{V}}$ such that $T_t \cup T_D \subseteq T'$, $T' \not\vdash_{\mathcal{L}_{\sqrt{V}}} \alpha_t$, $L_{T'}$ is a totally ordered $PMV_{\frac{1}{2^4}}$-algebra. Thus $[\alpha_t] \neq 1$. Consider the natural $Term_{\sqrt{V}}$-valuation $v : Term_{\sqrt{V}} \to L_{T'}$ i.e., $s \mapsto v(s) = [s]$. Then $[\alpha_t] = v(\alpha_t) \neq 1$. Moreover, by Theorem 7.6, $v(\beta) = 1$ for each $\beta \in T'$.

By Proposition 7.13 there exits a valuation $e : Term_{\sqrt{q\mathcal{PMV}}} \to D_{L_{T'}}$ such that $e_p(\beta) = v(\beta_t)$ for each $\beta \in Term_{\sqrt{q\mathcal{PMV}}}$. One the one hand, for each $\gamma \in T$, $e_p(\gamma) = v(\gamma_t) = 1$ since $\gamma_t \in T'$. Hence $e_p(T) = 1$. On the other hand, $e_p(\alpha) = v(\alpha_t) \neq 1$ which is a contradiction since $T \models^{Prob}_{\mathcal{IP}} \alpha$. Thus $T \vdash_{\mathcal{LIP}} \alpha$. For the converse see Proposition 7.3.

$\square$

Now we can establish a compactness theorem for the probabilistic consequence:

Theorem 7.15 *Let T be a theory and α be a term both in $Term_{\mathcal{IP}}$. Then:*

$$T \models^{Prob}_{\mathcal{IP}} \alpha \quad \textit{iff} \quad \exists\, T_0 \subseteq T \textit{ finite such that } T_0 \models^{Prob}_{\mathcal{IP}} \alpha$$

Proof: If $T \models^{Prob}_{\mathcal{IP}} \alpha$ by Theorem 7.14 there exists a proof of α, $\alpha_1, \cdots \alpha_n, \alpha$ from T. If we consider $T_0 = \{\alpha_k \in T : \alpha_k \in \{\alpha_1, \cdots \alpha_n\}\}$ then $T_0 \models^{Prob}_{\mathcal{IP}} \alpha$. The converse is immediate. $\square$

8 Conclusion

In this paper we have developed a logical-algebraic study for the system of quantum computational gates known as Poincaré irreversible quantum computational system or $\mathbb{IP}$-system for short. The $\mathbb{IP}$-system is interesting not only due to its relation with the continuous t-norms but also because it may be possibly applicable to the study of error-correcting codes [20] in the context of quantum computation. Several algebraic structures originated in reducts of the $\mathbb{IP}$-system, as qMV-algebras and $\sqrt{qMV}$-algebras, were introduced and studied in recent years, remaining as an open problem that posed in [3] and [5] about the axiomatizability of the $\mathbb{IP}$-system.

Facing this situation, the main results of this paper are the follwing: i) We have introduced an algebraic structure, the IP-algebra, that allows to give a mathematical representation of circuits made from assemblies of quantum gates of the $\mathbb{IP}$-system. ii) We have established a Hilbert-style calculus and a completeness theorem respect to the variety of IP-algebras, thus providing an answer to the mentioned open problem.

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